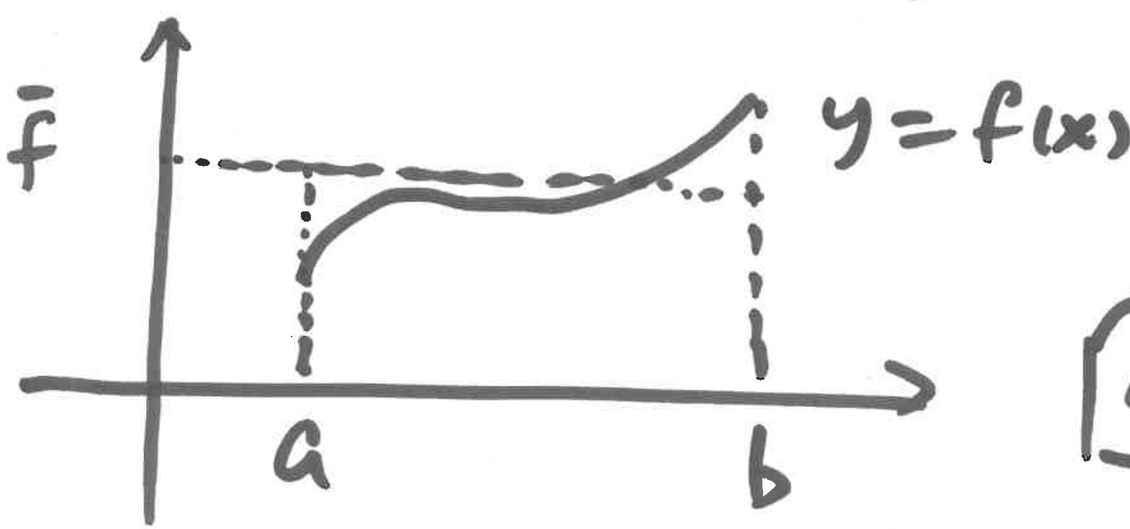
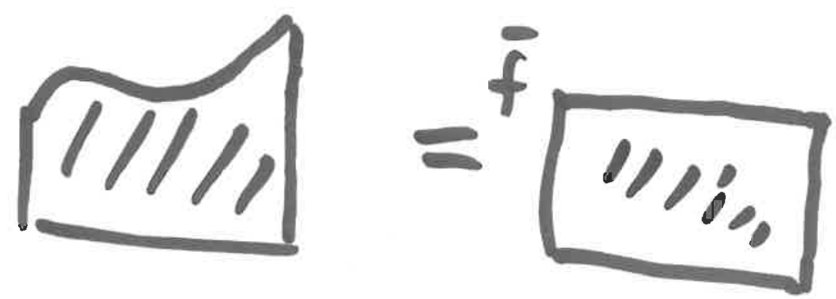
22.5 Average values

Def The <u>average value</u> of $f(x)$ on $[a, b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$





Ex 1 Compute the average value of $f(x)=\sqrt{x}$ on $[0,1]$ and $[0,2]$

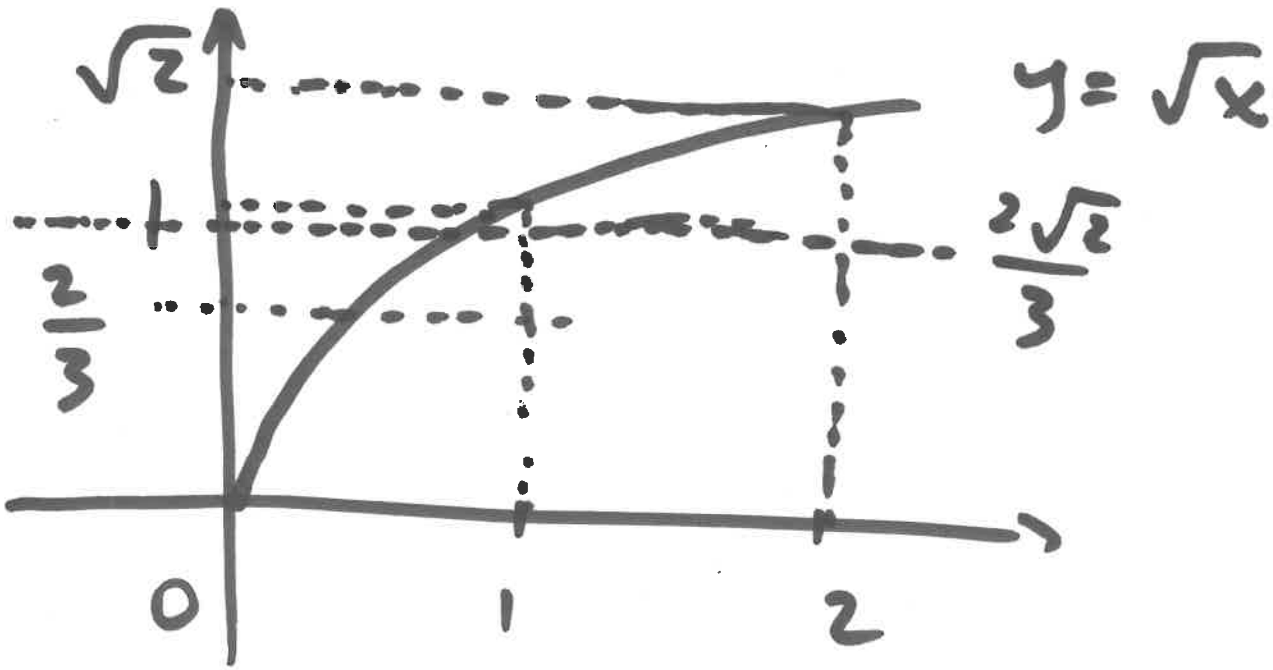
On $[0,1]$, $\bar{f} = \frac{1}{1-0}\int_0^1 \sqrt{x}\,dx$

$= \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}(1-0) = \frac{2}{3}$

On $[0,2]$, $\bar{f} = \frac{1}{2-0}\int_0^2 \sqrt{x}\,dx$

$= \frac{1}{2}\cdot\frac{2}{3}x^{\frac{3}{2}}\Big|_0^2 = \frac{1}{3}(2^{\frac{3}{2}}-0) = \frac{2\sqrt{2}}{3}$

$y = \sqrt{x} \qquad y^2 = x$



Ex 2 Let popu. $N(t)$ follow exp. growth. If initial popu. is 100 and at $t = 3$ popu. is 1000, find ave. popu. from $t = 0$ to $t = 3$

$$N(t) = N_0 e^{kt}$$

$\uparrow$ initial popu. $= 100$

$$1000 = N(3) = 100\, e^{k \cdot 3}$$

$$e^{3k} = 10$$

$$3k = \ln 10$$

$$k = \frac{\ln 10}{3}$$

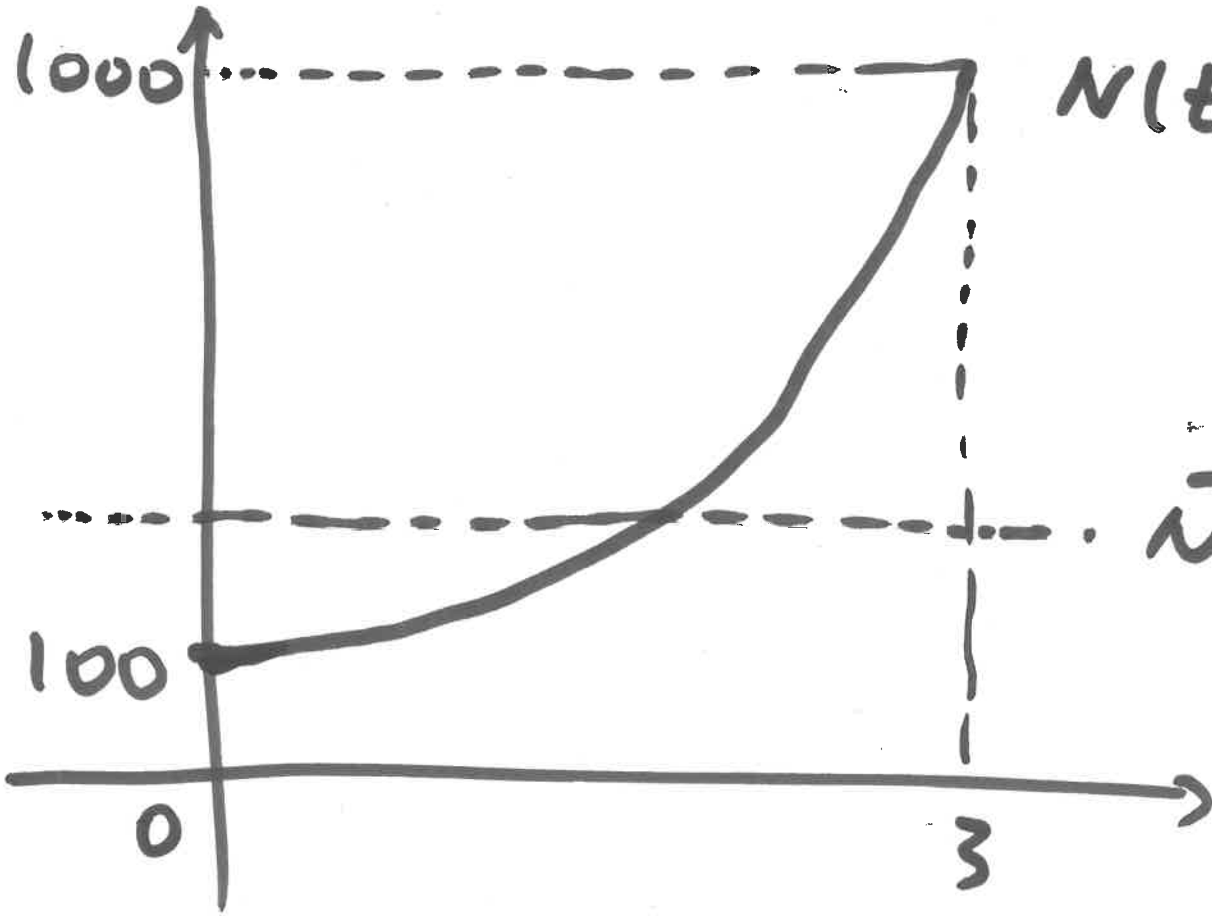
$$N(t) = 100\, e^{\frac{\ln 10}{3} t}$$

$$\bar{N} = \frac{1}{3-0}\int_0^3 100\, e^{\frac{\ln 10}{3}t}\, dt$$

$$= \frac{100}{3} \cdot \frac{1}{\frac{\ln 10}{3}} \cdot e^{\frac{\ln 10}{3}t}\Big|_0^3$$

$$= \frac{100}{\ln 10} \cdot \Big(\underbrace{e^{\frac{\ln 10}{3}\cdot 3}}_{=10} - \underbrace{e^0}_{=1} \Big)$$

$$= \frac{900}{\ln 10} \approx 391.$$



$$N(t) = 100\, e^{\frac{\ln 10}{3} t}$$

$$\bar{n} \approx 391$$

Ex 3 Let environment temperature in a day given by

$$f(t) = \begin{cases} 80 & 14 \le t \le 18 \\ 60 - 20\cos\left(\frac{\pi}{10}(t-4)\right) & 0 \le t < 14,\ 18 < t \le 24 \end{cases}$$

$(t : \text{hour},\ f : {}^\circ F)$

(1) Is $f(t)$ continuous on $[0, 24]$?

(2) Find ave. temp. during the whole day.

$$f(t) = \begin{cases} 80 & 14 \le t \le 18 \\ 60 - 20\cos\left(\frac{\pi}{10}(t-4)\right) & 0 \le t < 14 \\ 60 - 20\cos\left(\frac{\pi}{10}(t-8)\right) & 18 < t \le 24 \end{cases}$$

(1) $t = 14$

$$\lim_{t \to 14-} f(t) = 60 - 20\cos\left(\frac{\pi}{10}(14-4)\right)$$

$$= 60 + 20 = 80$$

$$\lim_{t \to 14+} f(t) = 80$$ //

$t = 18$

$$\lim_{t \to 18^-} f(t) = 80$$

$\parallel$

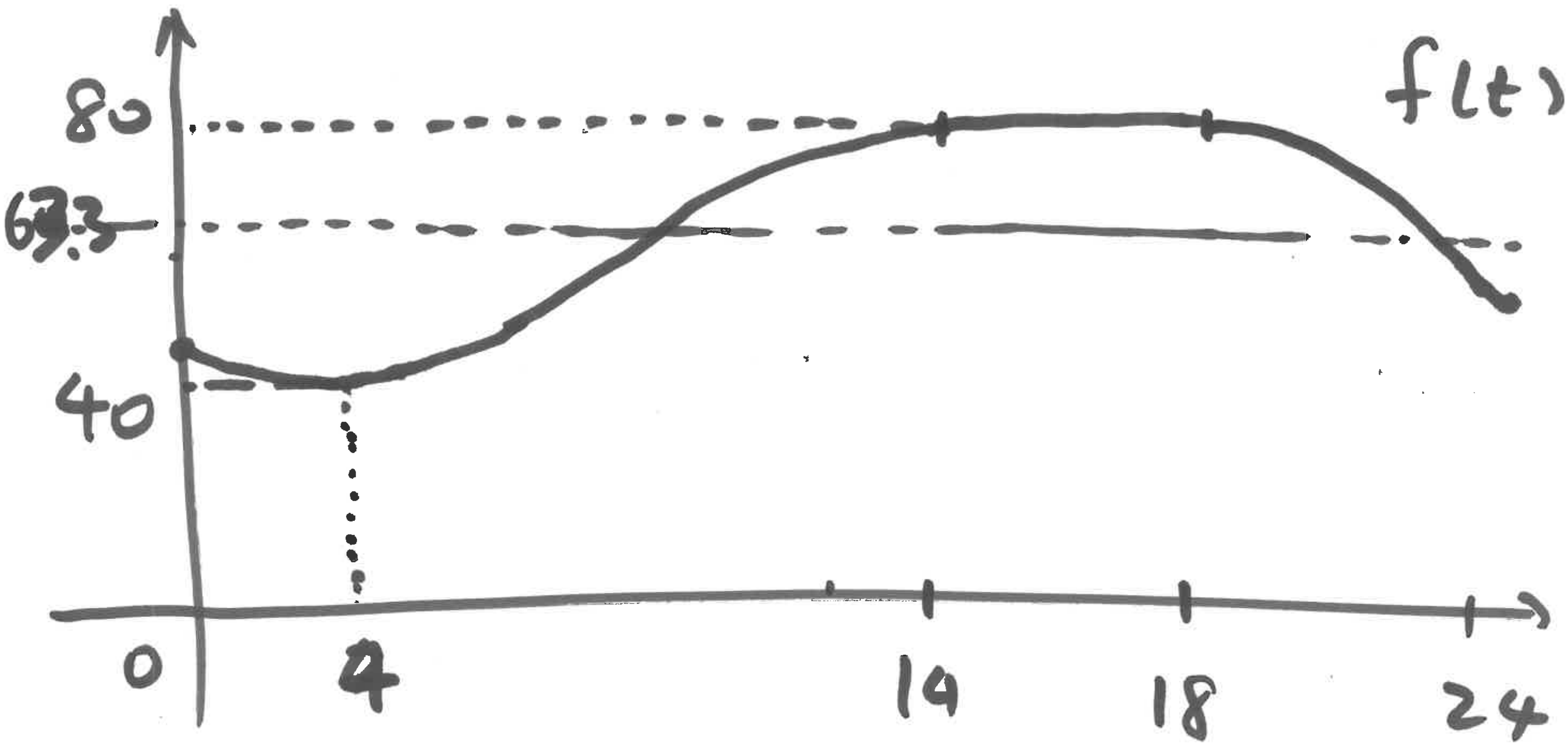
$$\lim_{t \to 18^+} f(t) = 60 - 20\cos\left(\frac{\pi}{10}(18-8)\right)$$

$$= 80$$

$\Rightarrow$ $f(t)$ is continuous

- Temperature functions should be continuous.

You can check that what I originally wrote there is not continuous at t=18, therefore I crossed it out because it is not a reasonable temperature model.



f(t)
80
67.3
40
0
4
14
18
24

$$\bar{f} = \frac{1}{24-0}\int_0^{24} f(t)\,dt$$

$$= \frac{1}{24}\left[\int_0^{14}\left(60 - 20\cos\left(\frac{\pi}{10}(t-4)\right)\right)dt\right.$$

$$+ \int_{14}^{18} 80\,dt$$

$$\left.+ \int_{18}^{24}\left(60 - 20\cos\left(\frac{\pi}{10}(t-8)\right)\right)dt\right]$$

- $\int \cos(ax+b)\,dx$

$$= \frac{1}{a}\sin(ax+b) + C$$

$$\int_{14}^{18} 80\,dt = 80\cdot t\Big|_{14}^{18}$$

$$= 80(18-14) = 320$$

$$\int_0^{14} \left(60 - 20\cos\left(\frac{\pi}{10}(t-4)\right)\right) dt$$

$$= \left(60t - 20\cdot\frac{10}{\pi}\sin\left(\frac{\pi}{10}(t-4)\right)\right)\Big|_0^{14}$$

$$= \left(60\cdot 14 - \frac{200}{\pi}\cdot 0\right) - \left(60\cdot 0 - \frac{200}{\pi}\sin\left(-\frac{4\pi}{10}\right)\right)$$

$$= 840 - \frac{200}{\pi}\sin\left(\frac{4\pi}{10}\right)$$

$$\int_{18}^{24} \cdots\cdots \, dt = 360 + \frac{200}{\pi}\sin\left(\frac{4\pi}{10}\right)$$

$$\bar{f} = \frac{1}{24}\Big(320 + 840 - \cancel{\frac{200}{\pi}\sin\frac{4\pi}{10}}$$

$$+ 360 + \cancel{\frac{200}{\pi}\sin\frac{4\pi}{10}} \Big)$$

$$\approx 63.3$$

Ex 4 The popu. growth rate $f(t)$ is given by $f(t) = e^{-t}$. Find ave. growth rate on $[0, 2]$, $[0, 10]$.

On $[0, 2]$, $\bar{f} = \frac{1}{2-0} \int_0^2 e^{-t}\,dt$

$$= \frac{1}{2} \cdot (-e^{-t}) \Big|_0^2$$

$$= \frac{1}{2}\left(-e^{-2} - (-e^{0})\right)$$

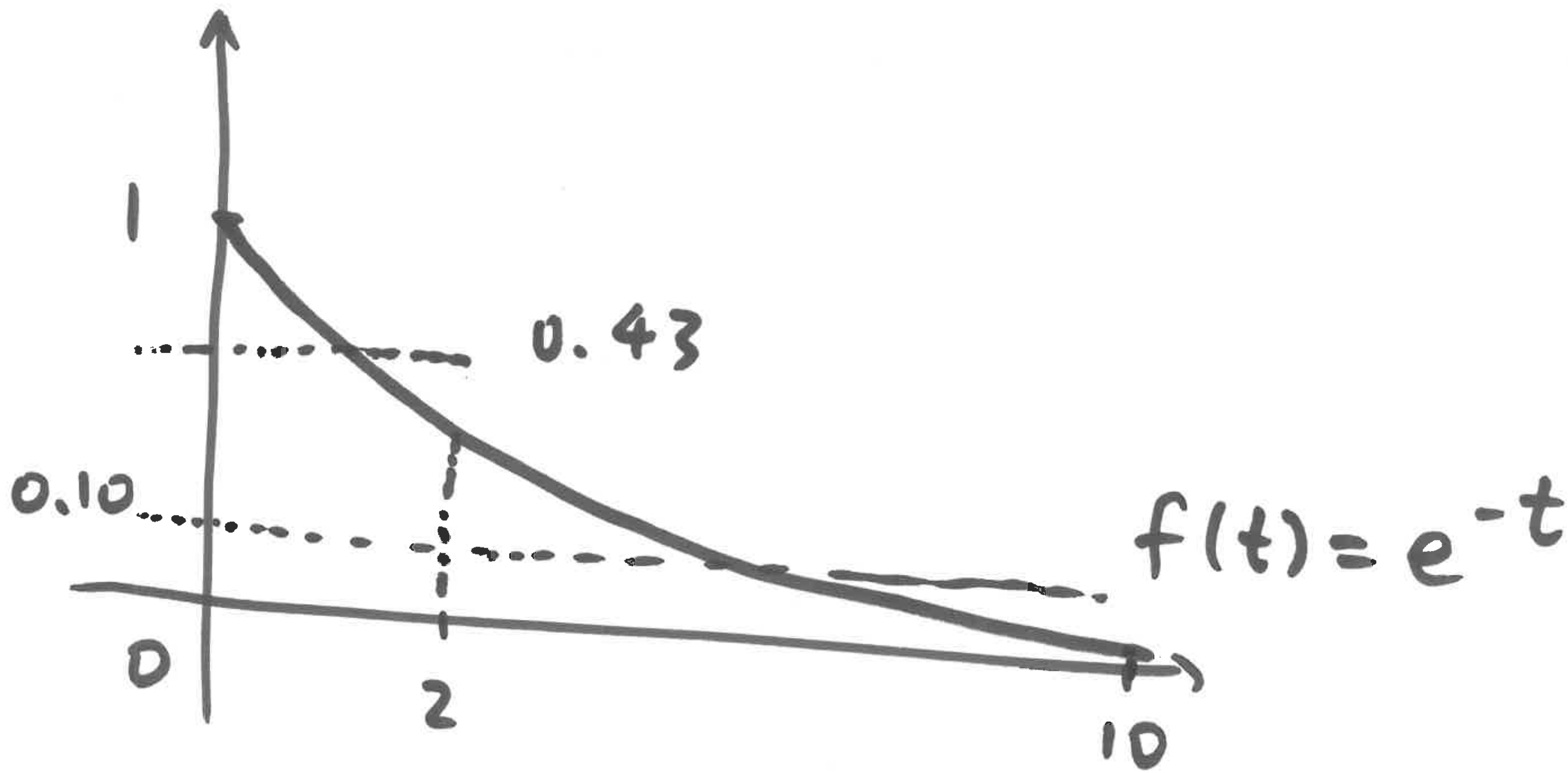
$$= \frac{1}{2}(-e^{-2} + 1) \approx 0.43$$

On $[0, 10]$, $\bar{f} = \dfrac{1}{10-0}\int_0^{10} e^{-t}\,dt$

$$= \frac{1}{10}\left(-e^{-t}\right)\Big|_0^{10}$$

$$= \frac{1}{10}\left(-e^{-10} - (-e^{0})\right)$$

$$= \frac{1}{10}\left(-e^{-10} + 1\right) \approx 0.10.$$



1
0.43
0.10
0
2
10
f(t) = e^{-t}

$N(t)$ popu.

$N'(t) = f(t)$ growth rate.

$$N(b) - N(a) = \int_a^b f(t)\,dt$$

$$\frac{N(b) - N(a)}{b-a} = \frac{1}{b-a}\int_a^b f(t)\,dt$$

If $f(t) = e^{-t}$ and $N(0) = 10$

$$N(T) - N(0) = \int_0^T e^{-t}\,dt$$

$$N(T) = 10 + \int_0^T e^{-t}\,dt$$

$$= 10 + (-e^{-t})\Big|_0^T$$

$$= 10 + (-e^{-T} - (-e^{0}))$$

$$= 11 - e^{-T}$$

ave. rate of change of $N(t)$ on $[0, 2]$.

$$\frac{N(2) - N(0)}{2 - 0} = \frac{(11 - e^{-2}) - (11 - e^{-0})}{2}$$

$$= \frac{11 - e^{-2} - 11 + 1}{2} = \frac{1}{2}(1 - e^{-2})$$

$N(t) = 11 - e^{-t}$