22.1 ~ 22.4 Anti-derivatives, integrals

Recall: fundamental theorem

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

if $F'(x) = f(x)$

Def An <u>anti-derivative</u> of $f(x)$ is a function $F(x)$ such that

$$F'(x) = f(x)$$

All the anti-derivatives of $f(x)$ (as a set) is called the <u>indefinite integral</u> of $f(x)$

- $\int f(x)\,dx = F(x) + C$

where F is an anti-der. of f, C is any constant.

<u>Ex</u> $f(x) = 1$ $\qquad \int f(x)\,dx = x + C$

Basic formulas:

$(x^n)' = nx^{n-1}$	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ $(n \neq -1)$
$(\ln x)' = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln x + C$ (for $x > 0$)
$(\sin x)' = \cos x$	$\int \cos x\,dx = \sin x + C$
$(\cos x)' = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$(e^{ax})' = ae^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$ $(a \neq 0)$

Ex 1 Compute indef. integrals

① $\int x^2\,dx = \frac{1}{3}x^3 + C$

② $\int \frac{2}{x^3}\,dx = 2\cdot\frac{1}{-2}x^{-2} + C = -\frac{1}{x^2} + C$

power $= -3$

③ $\int (x^3+1)^2\,dx = \int (x^6 + 2x^3 + 1)\,dx$

$= \frac{1}{7}x^7 + 2\cdot\frac{1}{4}x^4 + x + C$

④ $\int \frac{2-x}{x^2}\,dx$

$= \int \left(\frac{2}{x^2} - \frac{x}{x^2}\right) dx$

$= 2 \cdot \frac{1}{-1} \cdot x^{-1} - \ln x + C$

$= -\frac{2}{x} - \ln x + C$

⑤ $\int (e^{-3x} + \sin x)\,dx$

$= \frac{1}{-3} e^{-3x} - \cos x + C$

⑥ $$\int \frac{x^2 + xe^{2x} + 2x\cos x}{x}\,dx$$

$$= \int (x + e^{2x} + 2\cos x)\,dx$$

$$= \frac{1}{2}x^2 + \frac{1}{2}e^{2x} + 2\sin x + C$$

⑦ $$\int e^{2x}(e^x + e^{-x})\,dx$$

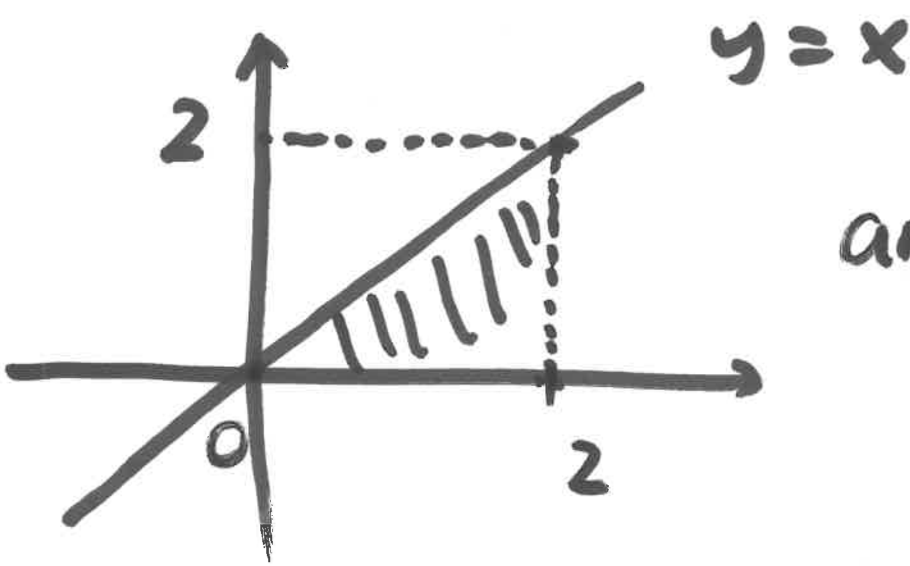
$$= \int (e^{3x} + e^x)\,dx$$

$$= \frac{1}{3}e^{3x} + e^x + C$$

__Ex 2__ Compute definite integrals:

① $\int_0^2 x\,dx = \frac{1}{2}x^2 \Big|_0^2 = \frac{1}{2}\cdot 2^2 - \frac{1}{2}\cdot 0^2 = \boxed{2}$

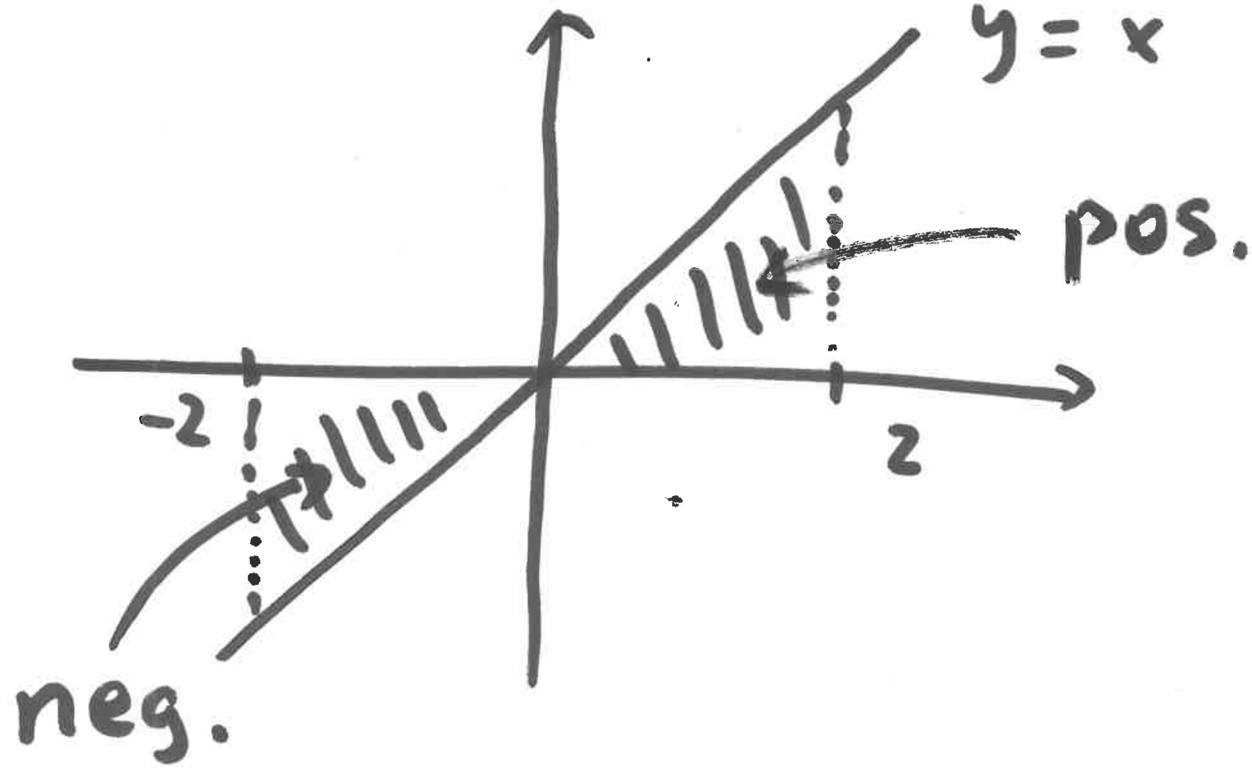
$F(x)\Big|_a^b$ means $F(b)-F(a)$.



area $= \frac{1}{2}\cdot 2\cdot 2 = 2$

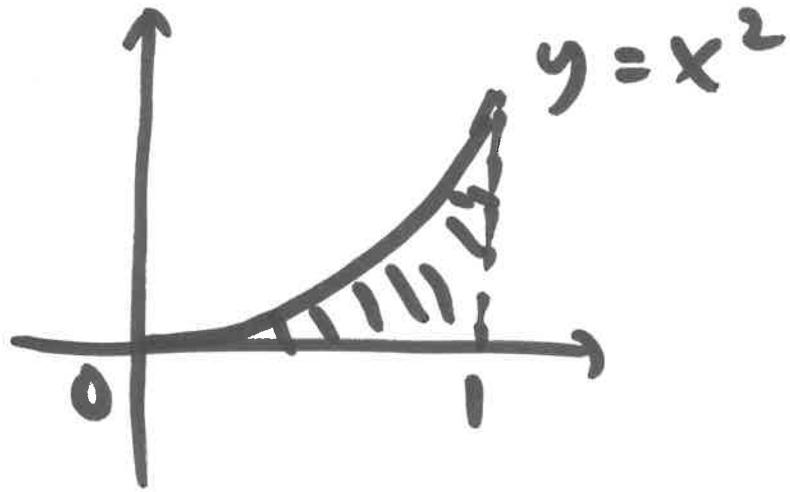
② $\int_{-2}^{2} x \, dx = \frac{1}{2}x^2 \Big|_{-2}^{2}$

$= \frac{1}{2} \cdot 2^2 - \frac{1}{2} \cdot (-2)^2 = 0$



③ $\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1$

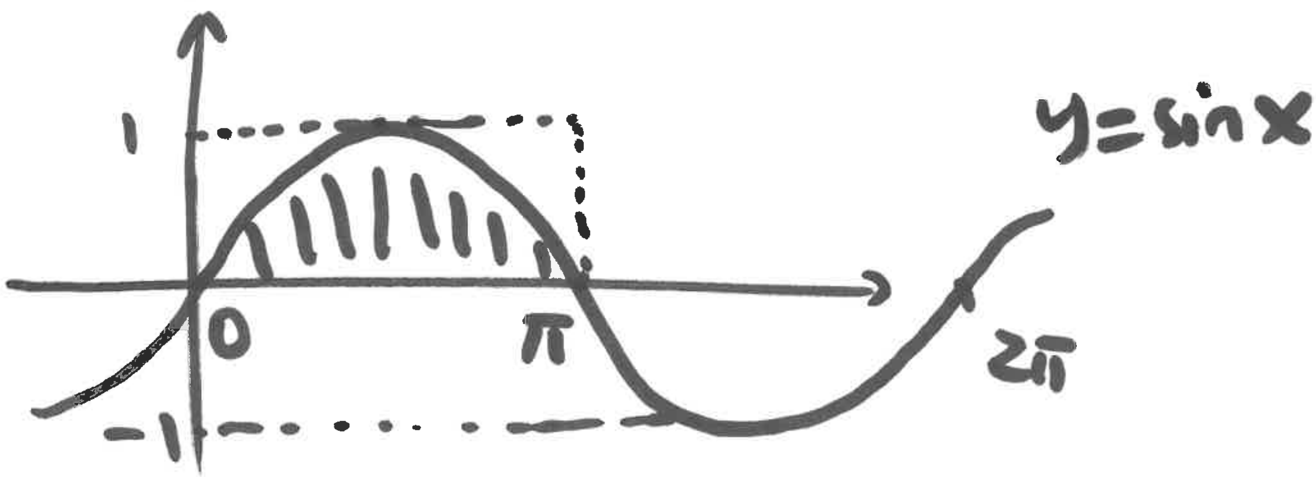
$= \frac{1}{3}\cdot 1^3 - \frac{1}{3}\cdot 0^3 = \frac{1}{3}$



④ $\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi}$

$= (-\cos \pi) - (-\cos 0)$

$= 1 - (-1) = 2$



$\int_0^{2\pi} \sin x \, dx = 0$

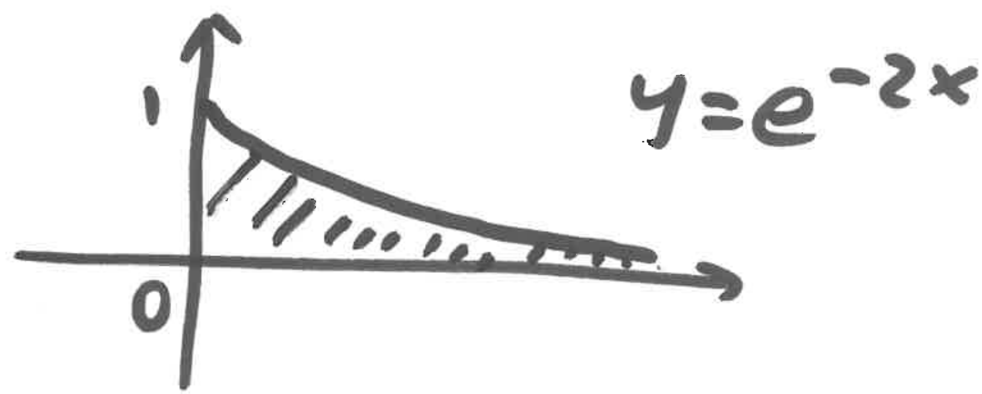
⑤ $\int_0^\infty e^{-2x}\,dx$

"improper integral" (not in exam)

$= \frac{1}{-2} e^{-2x} \Big|_0^\infty$

$= \frac{1}{-2} \lim_{x\to\infty} \underbrace{e^{-2x}}_{=0} - \frac{1}{-2} e^{-2\cdot 0}$

$= \frac{1}{2}$.



Ex 3 The growth rate of popu. of bacteria A, B are given by $f_A(t) = t$ and $f_B(t) = e^t$

If initially the popu. of A, B are 20 and 10, which type has larger popu. at $t=5$?

$$P_A'(t) = f_A(t).$$

$$\int_0^T \underbrace{f_A(t)}_{=t}\,dt = P_A(T) - P_A(0)$$

$$\int_0^T t\,dt = \frac{1}{2}t^2\Big|_0^T = \frac{1}{2}T^2 - \frac{1}{2}0^2$$

$$= \frac{1}{2}T^2$$

$$P_A(T) = P_A(0) + \frac{1}{2}T^2.$$

$$= 20 + \frac{1}{2}T^2$$

$$\int_0^T \underbrace{f_B(t)}_{\approx e^t} dt = P_B(T) - P_B(0)$$

$$\int_0^T e^t \, dt = e^t \Big|_0^T = e^T - e^0$$

$$= e^T - 1 .$$

$$P_B(T) = P_B(0) + e^T - 1$$

$$= 10 + e^T - 1$$

$$= 9 + e^T$$

$$P_A(5) = 20 + \frac{1}{2} \cdot 5^2 = 32.5$$

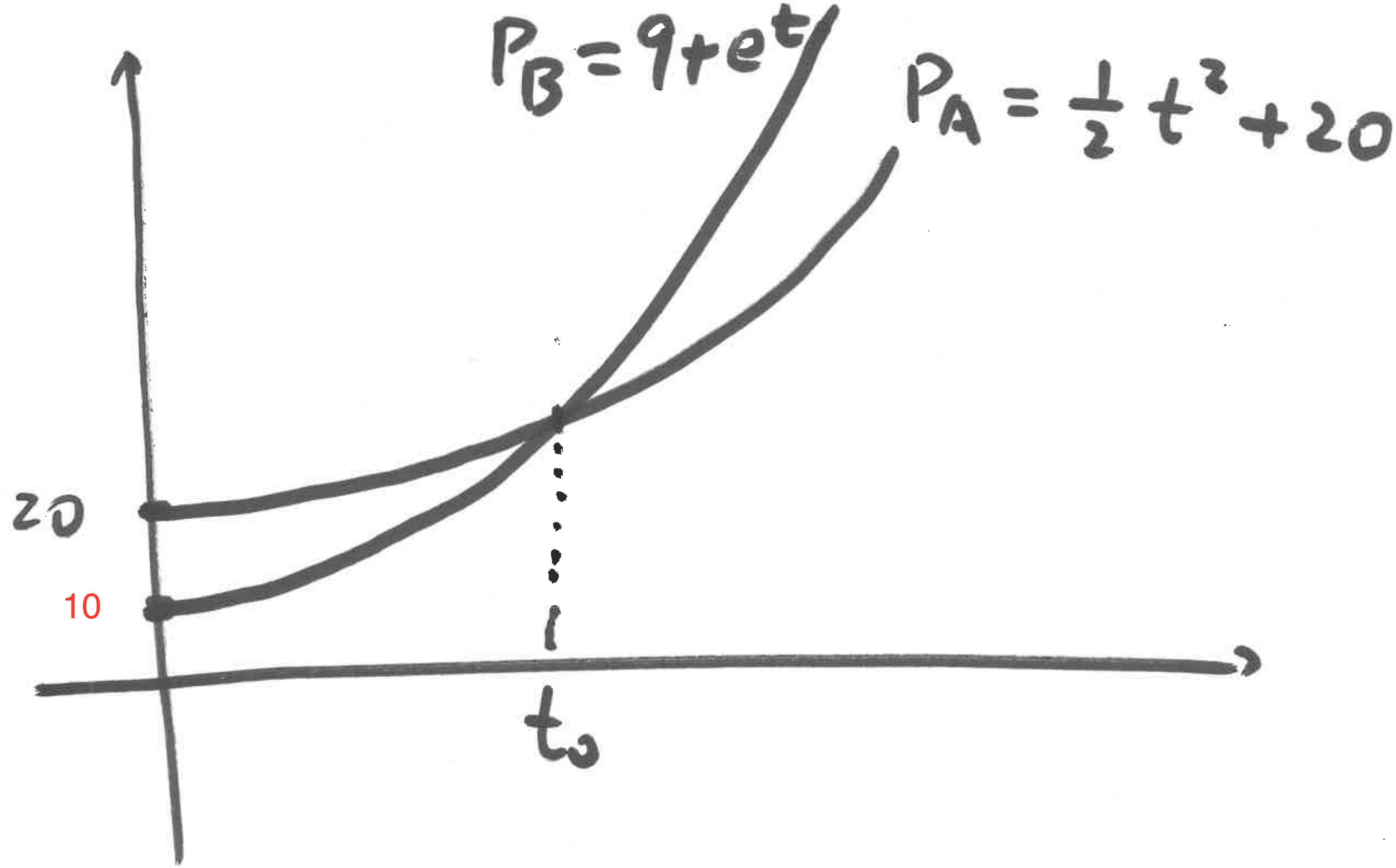
$$P_B(5) = 9 + e^5 \approx 157.4$$

when are the two popu. same ?

$$P_A(t) = P_B(t)$$

$$20 + \frac{1}{2} t^2 = 9 + e^t$$

$$e^t = \frac{1}{2} t^2 + 11 \Rightarrow t_0 = ?$$



$P_B = 9 + e^t$
$P_A = \frac{1}{2}t^2 + 20$
20
10
t_0