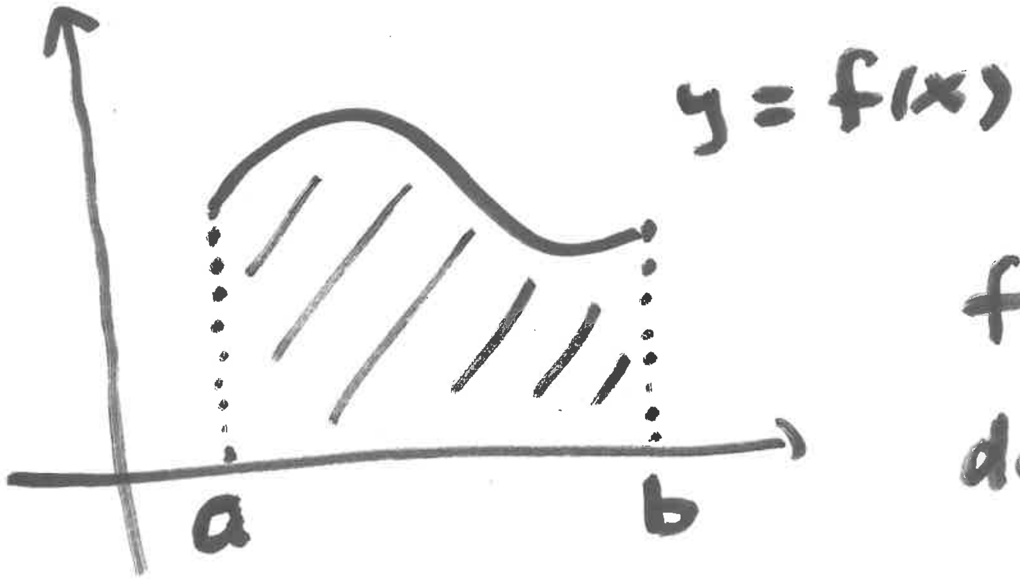
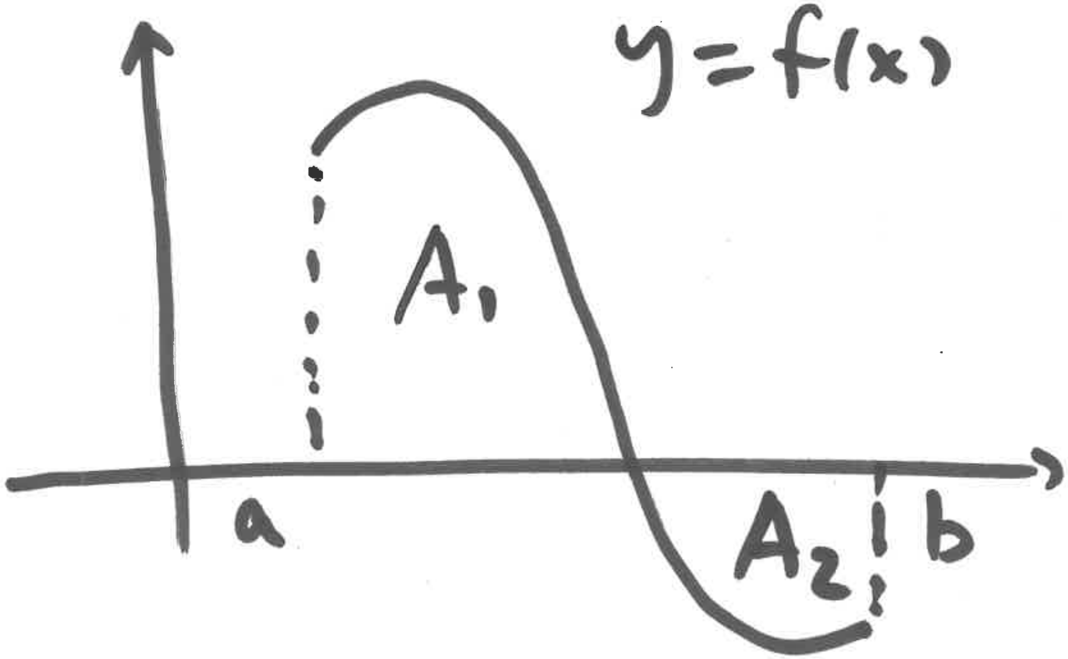
21.1 ~ 21.3 Area under curve, integral



Def For a function $f(x)$ defined on $[a,b]$ define the <u>integral</u> of f on $[a,b]$ to be the area of the region bounded by $y=f(x)$, $x=a$, $x=b$, $y=0$.

denoted as $\boxed{\int_a^b f(x)\,dx}$

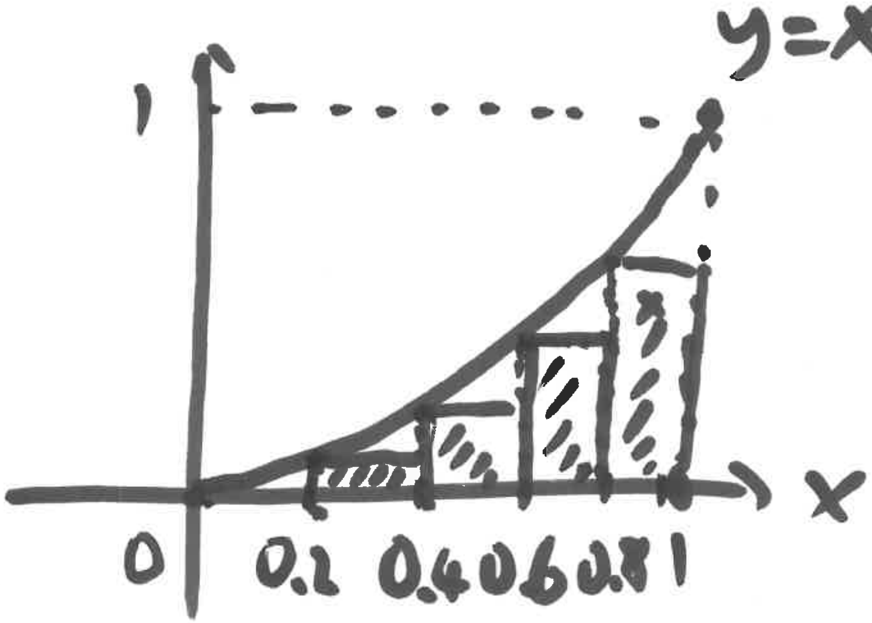


$$\int_a^b f(x)\,dx = A_1 - A_2$$

· Approximate integral by rectangles

Ex 1 $\int_0^1 x^2\,dx$

$f(x) = x^2$



$$\int_0^1 x^2\,dx$$

$$\approx \big(f(0)\cdot 0.2 + f(0.2)\cdot 0.2 + f(0.4)\cdot 0.2 + f(0.6)\cdot 0.2 + f(0.8)\cdot 0.2\big)$$

$$= 0.2\cdot(0 + 0.04 + 0.16 + 0.36 + 0.64)$$

$$= 0.24$$

$\Delta x = 0.2$

$f(0) = 0$

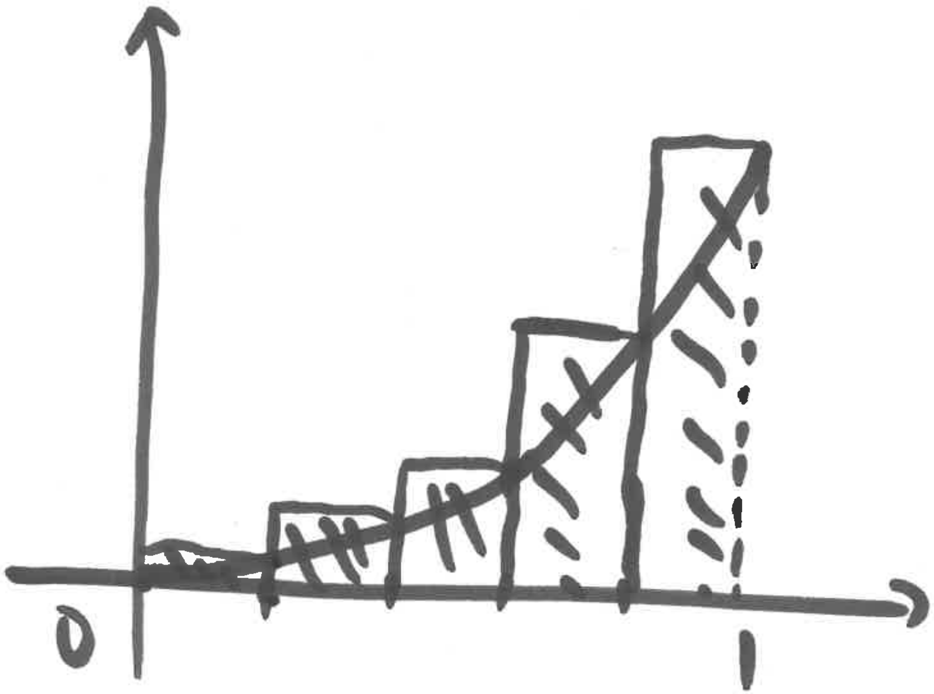
$f(0.2) = 0.04$

$f(0.4) = 0.16$

$f(0.6) = 0.36$

$f(0.8) = 0.64$

$f(1) = 1$



$$\int_0^1 x^2\,dx$$

$$\approx 0.2\cdot\left(f(0.2)+f(0.4)+f(0.6)+f(0.8)+f(1)\right)$$

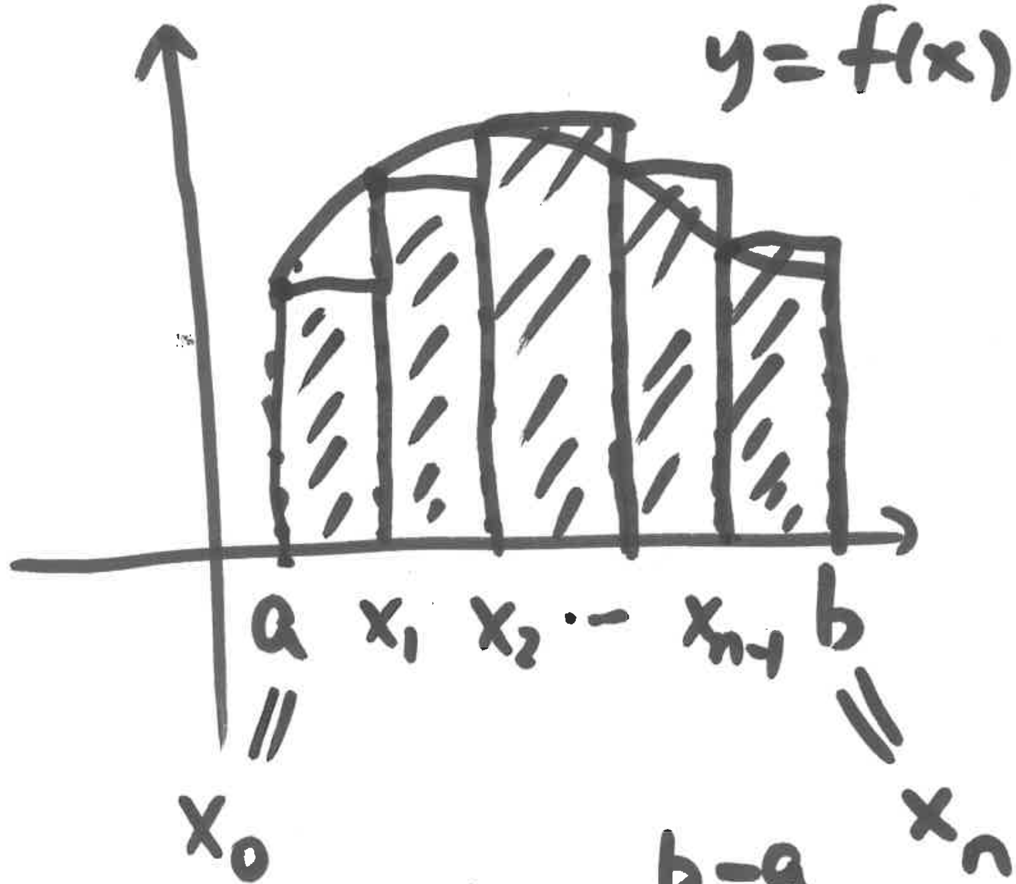
$$= 0.2\cdot\left(0.04+0.16+0.36+0.64+1\right)$$

$$= 0.44$$

$$0.24 < \int_0^1 x^2\,dx < 0.44$$

In fact, $\int_0^1 x^2\,dx = \frac{1}{3}$

More generally. $\int_a^b f(x)\,dx$



$$\approx \Delta x \cdot \left(f(x_0) + f(x_1) + \cdots + f(x_{n-1}) \right)$$

$$= \Delta x \cdot \sum_{i=0}^{n-1} f(x_i)$$

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\,\Delta x$$

$$\int_a^b f(x)\,dx \approx \Delta x \cdot \sum_{i=0}^{n-1} f(x_i)$$

$$\Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x$$

$$\int_a^b f(x)\,dx \approx \Delta x \cdot \sum_{i=1}^{n} f(x_i)$$

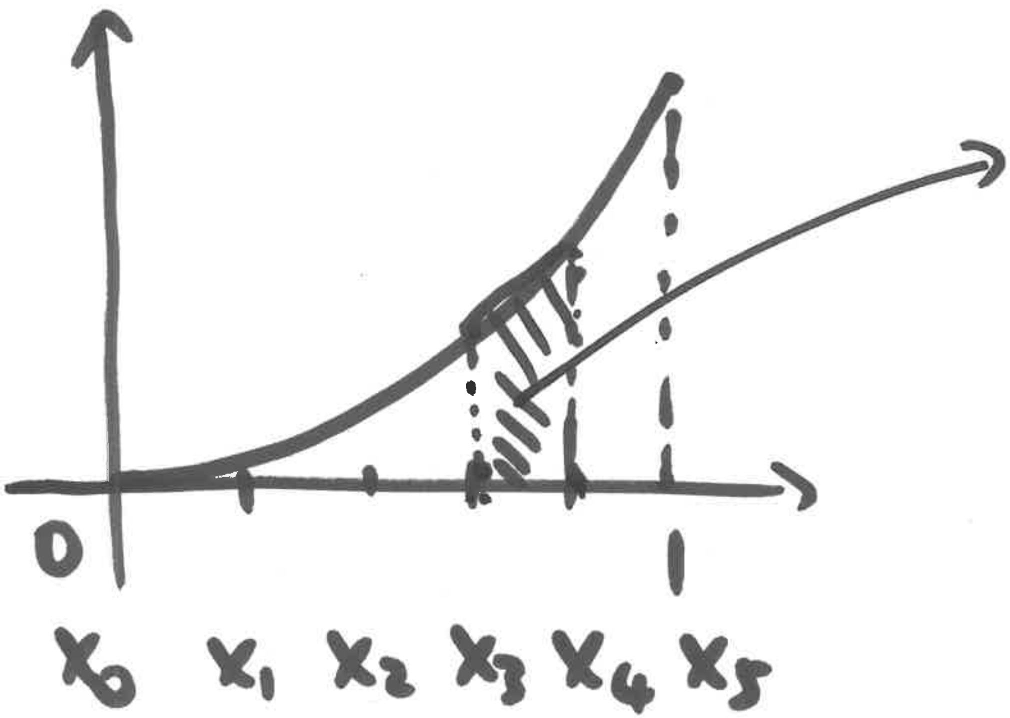
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{i=0}^{n-1} f(x_i)$$

$$= \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i)$$

This is the rigorous definition !

- Approximate integral by trapezoid.

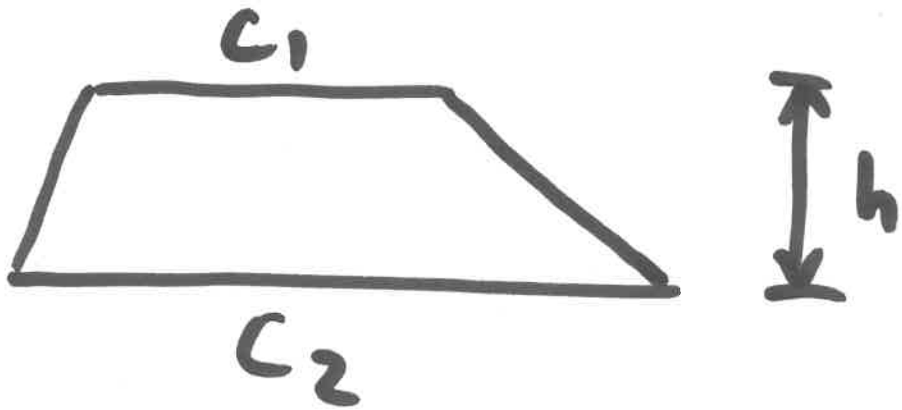
Ex 2 $\int_0^1 x^2\,dx$



$$\text{area} = \frac{1}{2}\cdot\left(f(x_3) + f(x_4)\right)\cdot\Delta x$$

$\Delta x = 0.2$

trapezoid



$$A = \frac{1}{2}(c_1 + c_2) \cdot h$$

$$\int_0^1 x^2\,dx \approx \frac{1}{2}\Delta x\cdot\Big[\big(f(x_0)+f(x_1)\big)$$

$$+\big(f(x_1)+f(x_2)\big)+\cdots\cdots+\big(f(x_{n-1})+f(x_n)\big)\Big]$$

$$=\frac{1}{2}\Delta x\cdot\Big[f(x_0)+2\big(f(x_1)+f(x_2)$$

$$+\cdots+f(x_{n-1})\big)+f(x_n)\Big]$$

$$=\frac{1}{2}0.2\cdot\Big[0+2\big(0.04+0.16+0.36$$

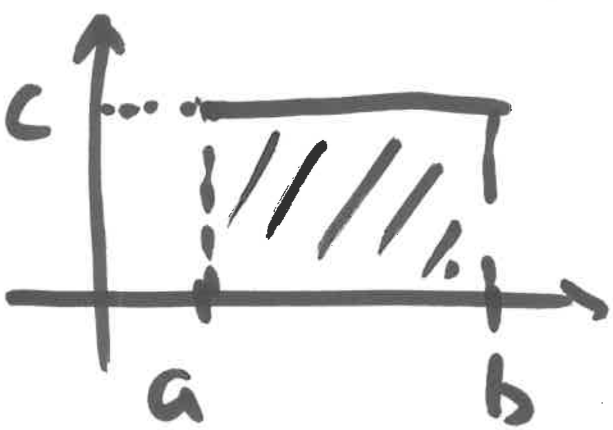
$$+0.64\big)+1\Big]=0.34$$

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\Delta x \cdot \left[f(x_0) + 2\left(f(x_1) + \cdots + f(x_{n-1})\right) + f(x_n) \right]$$

- Suppose $f(t)$ is the rate of change of $F(t)$. (in other words, $F'(t) = f(t)$).

$$\int_a^b f(t)\,dt = ?$$

① When $f(t) = c$

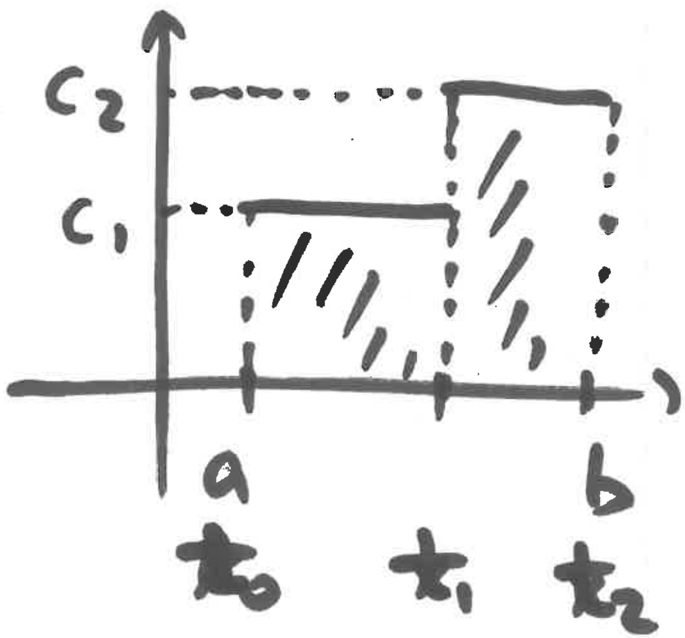


$$\int_a^b f(t)\,dt = c\cdot(b-a)$$

= change in F from a to b.

$= F(b) - F(a)$

② When $f(t) = \begin{cases} c_1 & a = t_0 < t < t_1 \\ c_2 & t_1 < t < t_2 = b \end{cases}$



$$\int_a^b f(t)\,dt$$

$$= c_1 \cdot (t_1 - t_0) + c_2 \cdot (t_2 - t_1)$$

$$= (F(t_1) - F(t_0)) + (F(t_2) - F(t_1))$$

$$= F(t_2) - F(t_0)$$

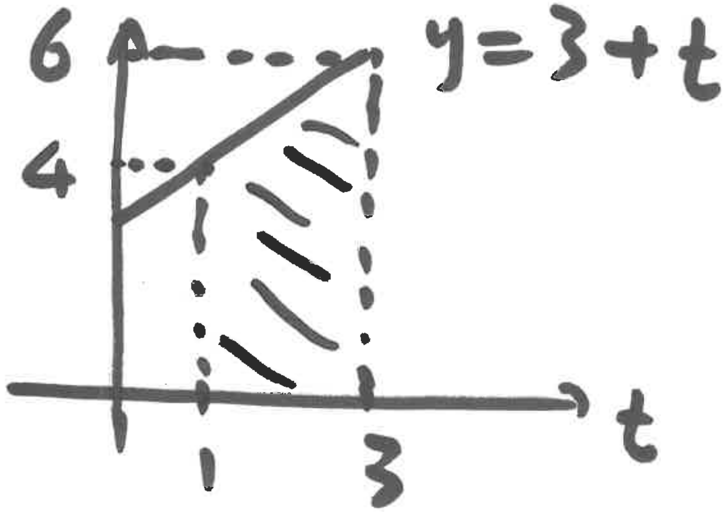
$$= F(b) - F(a).$$

$$\boxed{\begin{aligned}&\int_a^b f(t)\,dt = F(b) - F(a)\\&\text{where } F'(t) = f(t)\end{aligned}}$$

"Fundamental theorem of calculus"

Ex 3 Suppose $f(t) = 3 + t$ is the rate of change of population $P(t)$. If the popu. is 10 at $t=1$, what is the popu. at $t=3$?

$$P(3) - P(1) = \int_1^3 (3+t)\,dt$$



$$= \frac{1}{2}(4+6)\cdot 2 = 10$$

$$P(1) = 10$$

$$P(3) = 10 + 10 = 20.$$

Ex 4 The velocity of a falling object is $v(t) = -10t$. If its initial height is 20, how long does it take to reach the ground?

height $h(t)$ $\qquad$ $h'(t) = v(t)$.

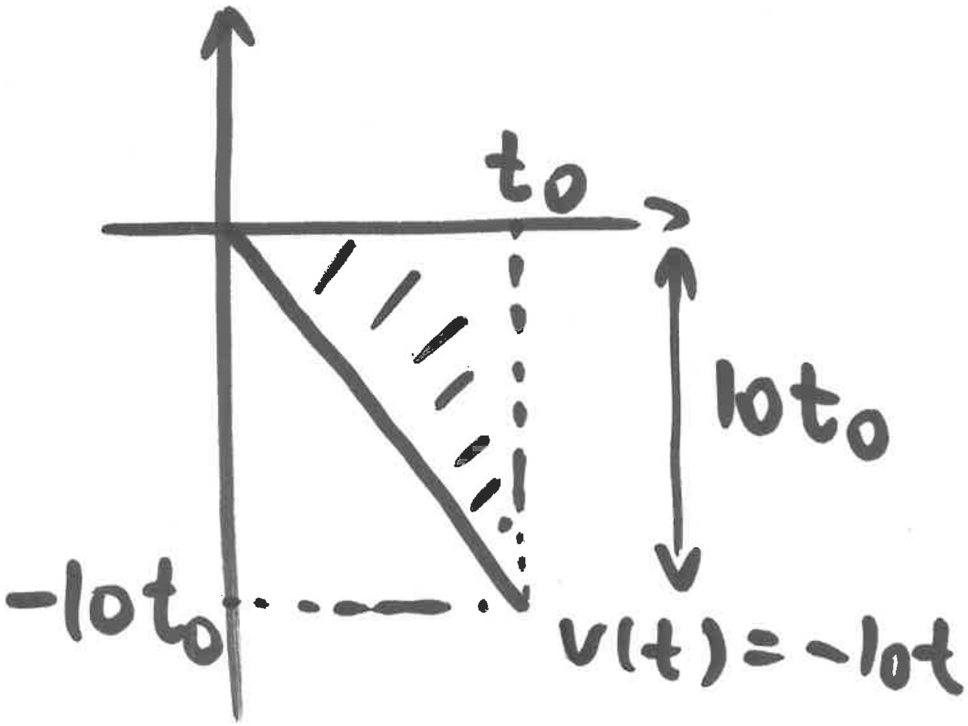
Know $h(0) = 20$

want t_0 such that $h(t_0) = 0$.

$$h(t_0) - h(0) = \int_0^{t_0} v(t)\,dt$$

$$= 0 - 20 = -20$$

$$= \int_0^{t_0} (-10t)\,dt = -\frac{1}{2} \cdot t_0 \cdot 10t_0 = -5t_0^2$$



$$-20 = -5t_0^2$$

$$t_0^2 = 4 \qquad t_0 = 2$$

$$t_0 \geq 0$$