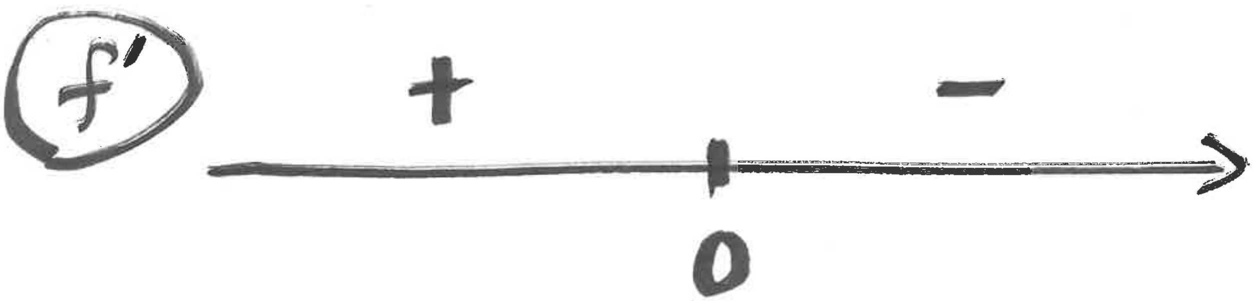
2a4 (continued)

Ex Sketch graph:

$$f(x) = e^{-x^2}$$

$$f'(x) = e^{-x^2} \cdot (-2x)$$

$$f'(x) = 0 \Rightarrow x = 0$$



$$f''(x) = \left(e^{-x^2}\right)' \cdot (-2x) + e^{-x^2} \cdot (-2x)'$$

$$= e^{-x^2} \cdot (-2x) \cdot (-2x) + e^{-x^2} \cdot (-2)$$

$$= e^{-x^2}\left(4x^2 - 2\right)$$

$$f''(x) = 0 \Rightarrow \quad 4x^2 - 2 = 0$$

$$4x^2 = 2$$

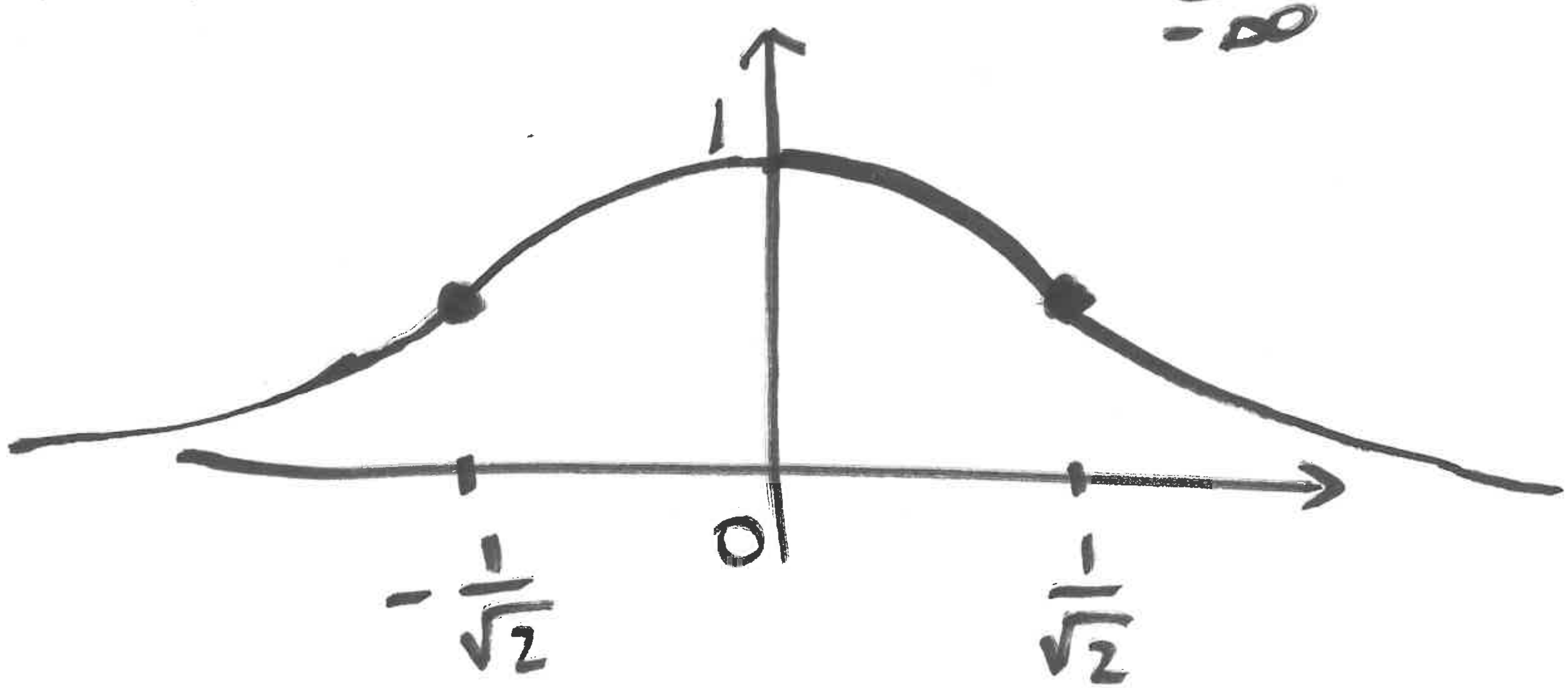
$$x^2 = \frac{1}{2}$$

$$x = \pm \frac{1}{\sqrt{2}}$$

f''

+		−		+
	$-\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	

$$\lim_{x \to \infty} e^{-x^2} = 0 \qquad \lim_{x \to -\infty} e^{-x^2} = 0$$

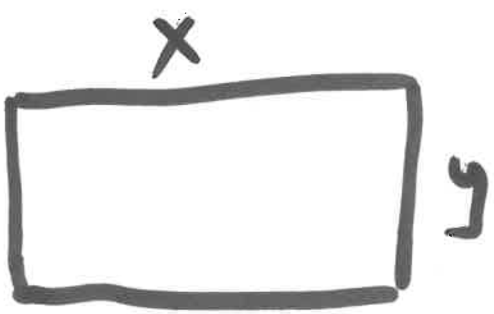


20.5 Optimization

- Identify the function to optimize
- Write it as a function of <u>one</u> variable
- Take derivative, find critical pts
- Check values of func. at crit. pts, boundary pts.

Ex 1 A rectangle has circumference equal to 2. What's the largest possible area?

Want to maximize the area



$$A = xy.$$

$$C = 2x + 2y = 2$$

$$2y = 2 - 2x$$

$$y = 1 - x$$

$$A(x) = x(1-x) = x - x^2$$

$$A'(x) = 1 - 2x$$

$$A'(x) = 0 \Rightarrow \quad 2x = 1 \qquad x = \frac{1}{2}$$

Domain: $x \geq 0 \qquad y \geq 0$

$$1 - x \geq 0$$

$$x \leq 1$$

$$[0, 1]$$

$$A\left(\frac{1}{2}\right) = \frac{1}{4}, \quad A(0) = 0, \quad A(1) = 0$$

↑

max

The max. area is $\frac{1}{4}$, achieved when $x = \frac{1}{2}$ (square shape)

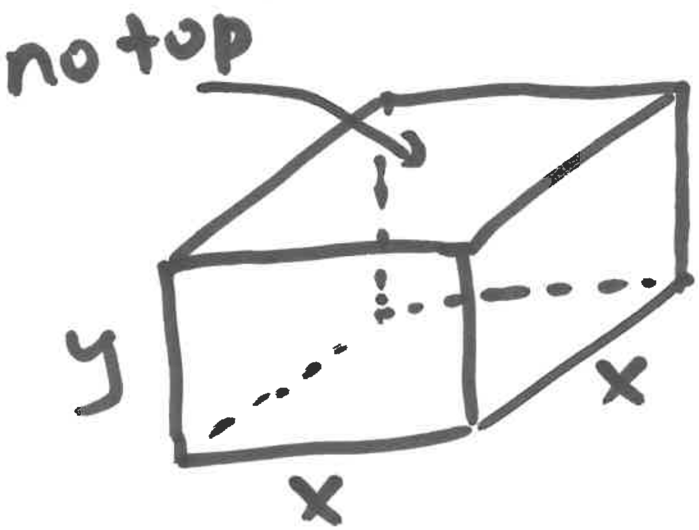
$\rightsquigarrow y = 1 - x = \frac{1}{2}$

$A''(x) = -2$

$A''\left(\frac{1}{2}\right) = -2 < 0$

(local max.)

Ex 2 Make a box of volume = 4 with square base, no top. What's the least amount of material to use?



Want to minimize

$$A = x^2 + 4xy$$

$$V = x^2 y = 4$$

$$y = \frac{4}{x^2}$$

$$A(x) = x^2 + 4x \cdot \frac{4}{x^2} = x^2 + \frac{16}{x}$$

$$A'(x) = 2x + 16 \cdot (-1) \cdot x^{-2}$$

$$= 2x - \frac{16}{x^2}$$

$$A'(x) = 0$$

$$2x = \frac{16}{x^2}$$

$$x^3 = 8$$

$$x = 2$$

Domain : $x > 0$

$$(0, \infty)$$

$$A(2) = 2^2 + \frac{16}{2} = 12 \leftarrow \text{min.}$$

$$\lim_{x \to 0+} \left(x^2 + \frac{16}{x} \right) = \infty$$

$$\lim_{x \to \infty} \left(x^2 + \frac{16}{x} \right) = \infty$$

The min. area is 12, achieved when $x = 2 \rightsquigarrow y = 1$

Ex 3 (logistic growth)

The population $N(t)$ satisfies

$$N(t) = \frac{K N_0}{N_0 + (K - N_0) e^{-rt}}$$

K, N_0, r positive constants

N_0 much smaller than K.

What's the time at which the popu. growth rate is largest?

Want to maximize

$$f(t) = N'(t)$$

$$= K N_0 \cdot \left(- \frac{1 \cdot (K - N_0)(-r)e^{-rt}}{(N_0 + (K - N_0)e^{-rt})^2} \right)$$

$$= K N_0 (K - N_0) r \cdot \frac{e^{-rt}}{(N_0 + (K - N_0)e^{-rt})^2}$$

$$f'(t) = K N_0 (K - N_0) r^2 e^{-rt} \cdot \frac{-N_0 + (K - N_0)e^{-rt}}{(N_0 + (K - N_0)e^{-rt})^3}$$

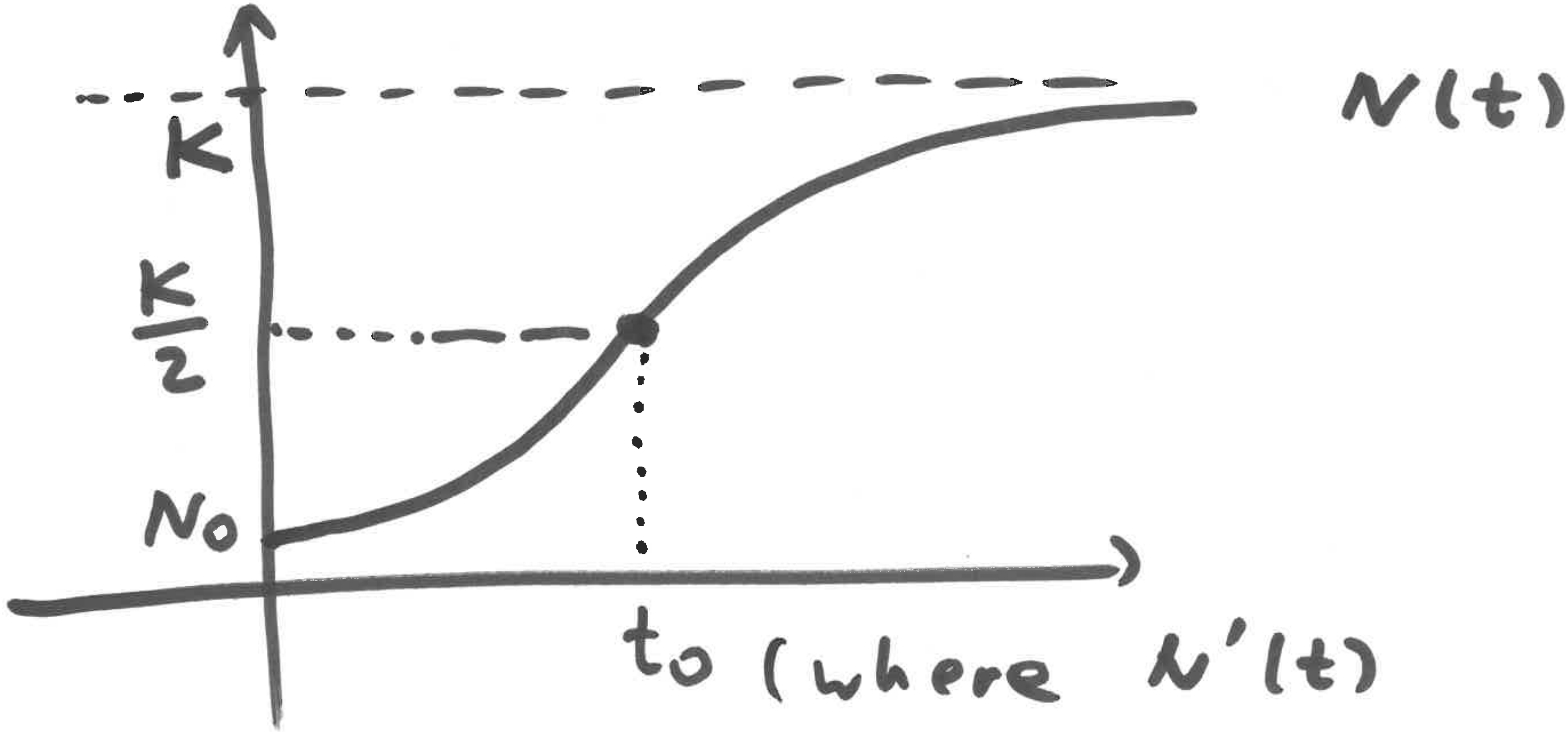
$$f'(t) = 0 \Rightarrow -N_0 + (K - N_0)e^{-rt} = 0$$

$$(K - N_0)e^{-rt} = N_0$$

$$e^{-rt} = \frac{N_0}{K - N_0}$$

$$-rt = \ln \frac{N_0}{K - N_0}$$

$$t_0 = -\frac{1}{r} \ln \frac{N_0}{K - N_0}$$



(where $N'(t)$ is maximized)

What is $N(t_0)$?

$$N(t_0) = \frac{K N_0}{N_0 + N_0} = \frac{K}{2}$$