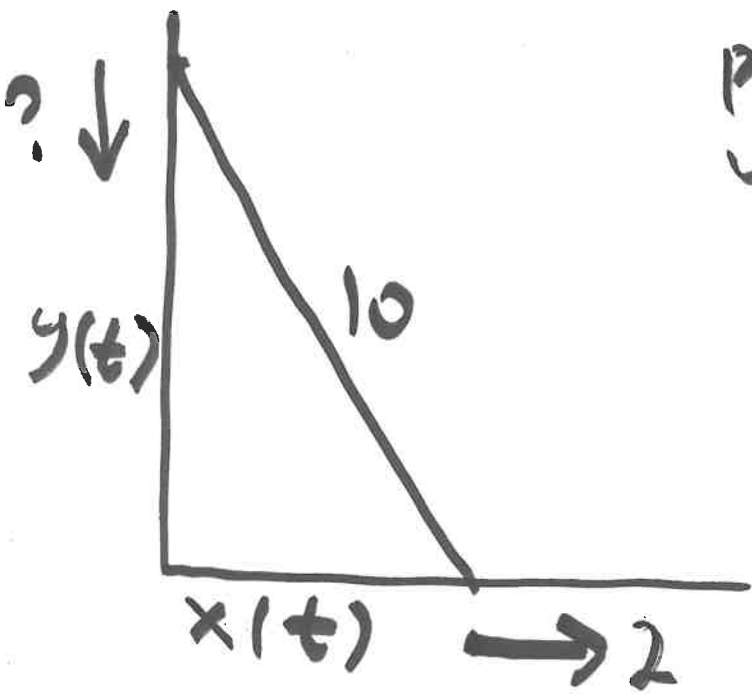
28.3 Related rates

- When two functions are related by an equation, find the relation between their derivates (by differentiating the equation).

Ex 1 A rod of length 10m leans on a wall corner, and it's sliding down. At $t=1\,s$, the bottom is 6 m from the corner, and moving at 2m/s. What is the speed of the top.



Pythagoras thm

$$\boxed{x(t)}^2 + y(t)^2 = 10^2.$$

$x(1) = 6 \qquad x'(1) = 2$

$y'(1) = ?$

$$2x(t)\cdot x'(t) + 2y(t)\cdot y'(t) = 0$$

$$x(1)^2 + y(1)^2 = 100$$

$$6^2 + y(1)^2 = 100$$

$$y(1)^2 = 100 - 36 = 64$$

$$y(1) = 8 \leftarrow y(t) > 0$$

$$2\cdot 6\cdot 2 + 2\cdot 8\cdot y'(1) = 0$$

$$y'(1) = -\frac{2\cdot 6\cdot 2}{2\cdot 8} = -\frac{3}{2}$$

Ex 2 A spherical bacteria is growing.
At $t = t_0$ its volume is $0.1\ mm^3$
and increasing at a rate of $0.001\ mm^3/s$

① What is the rate of change of its radius at $t = t_0$?

② ..... surface area .... ?

$$V = \frac{4}{3}\pi R^3$$

$$V(t) = \frac{4}{3}\pi \boxed{R(t)}^3$$

$$V(t_0) = 0.1 \quad , \quad V'(t_0) = 0.001$$

$$R'(t_0) = ?$$

$$V'(t) = \frac{4}{3}\pi \cdot 3\, R(t)^2 \cdot R'(t).$$

$$V(t_0) = \frac{4}{3}\pi R(t_0)^3$$

$$\| \; 0.1$$

$$R(t_0)^3 = \frac{0.1}{\frac{4}{3}\pi} = \frac{0.3}{4\pi}$$

$$R(t_0) = \sqrt[3]{\frac{0.3}{4\pi}} \simeq 0.288$$

$$V'(t_0) = \frac{4}{3}\pi \cdot 3R(t_0)^2 \cdot R'(t_0)$$

$$R'(t_0) = \frac{V'(t_0)}{\frac{4}{\cancel{3}}\pi \cdot \cancel{3}R(t_0)^2}$$

$$= \frac{0.001}{4\pi \cdot 0.288^2} \approx 9.59 \times 10^{-4} \text{ mm/s}$$

$$S(t) = 4\pi \left[R(t)\right]^2$$

$$S'(t) = 4\pi \cdot 2R(t) \cdot R'(t)$$

$$S'(t_0) = 8\pi \cdot R(t_0) \cdot R'(t_0)$$

$$= 8\pi \cdot 0.288 \cdot 9.59 \times 10^{-4}$$

$$\approx 0.00694 \ mm^2/s$$

Ex 3 A cylindrical bacteria is changing shape while keeping volume at $2\,mm^3$. At $t=2s$, its base radius is 0.3 mm and increasing at a rate of 0.01 mm/s

① How fast is its height changing? at $t=2s$

② . . . . Surface area . . ?

$$V = \pi R^2 h$$

$$2 = \pi \underline{R(t)^2} \cdot \underline{h(t)}$$

$$R(2) = 0.3 \quad , \quad R'(2) = 0.01$$

$$h'(2) = ?$$

$$0 = \pi \cdot \left(\left(R(t)^2\right)' \cdot h(t) + R(t)^2 \cdot h'(t) \right)$$

$$= \pi \left(2R(t) \cdot R'(t) \cdot h(t) + R(t)^2 h'(t) \right)$$

$$2 = \pi\, R(2)^2 \cdot h(2)$$

$$h(2) = \frac{2}{\pi \cdot 0.3^2} \approx 7.07$$

$$2\,\cancel{R(2)} \cdot R'(2) \cdot h(2) + R(2)^{\cancel{2}}\, h'(2) = 0$$

$$R(2)\, h'(2) = -2R'(2)\, h(2)$$

$$h'(2) = -\frac{2R'(2)\,h(2)}{R(2)}$$

$$= -\frac{2 \cdot 0.01 \cdot 7.07}{0.3} \approx -0.471 \text{ mm/s}$$

$$S(t) = 2\pi R(t)^2 + 2\pi R(t)\, h(t)$$

$$S'(t) = 2\pi \cdot 2R(t) \cdot R'(t)$$

$$+ 2\pi \cdot \left(R'(t)\, h(t) + R(t)\, h'(t) \right)$$

$$S'(2) = 2\pi \cdot 2 \cdot 0.3 \cdot 0.01$$

$$+ 2\pi \left(0.01 \cdot 7.07 + 0.3 \cdot (-0.471) \right)$$

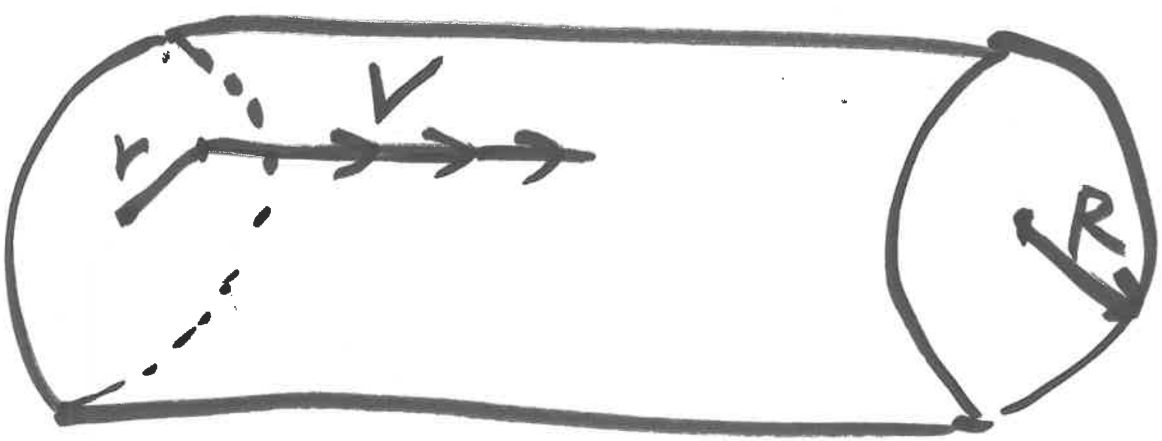
$$= -0.406\ \mathrm{mm}^2/\mathrm{s}$$

Ex 4 Blood flow velocity

$$V = k(R^2 - r^2) \qquad k = 375$$

R: radius of vessel

r: distance from the center.



At $t = t_0$, the vessel radius is 0.06 mm, and increasing at a rate of 0.01 mm/min, how fast is blood velocity changing?

$$V(t) = k\left(R(t)^2 - r^2\right)$$

$$R(t_0) = 0.06, \quad R'(t) = 0.01$$

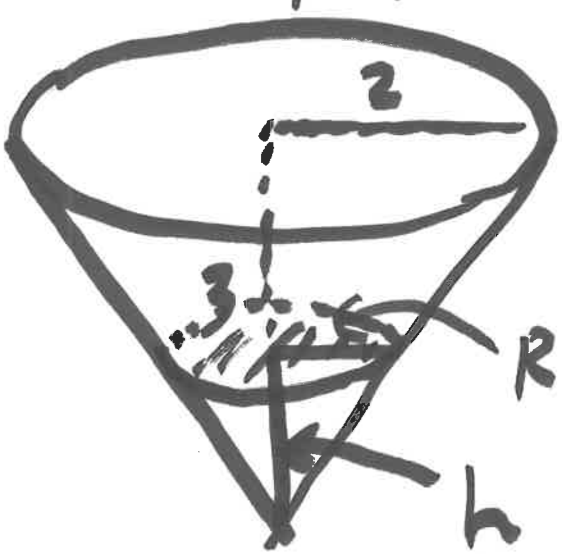
$$V'(t_0) = ?$$

$$V'(t) = k \cdot 2R(t) \cdot R'(t)$$

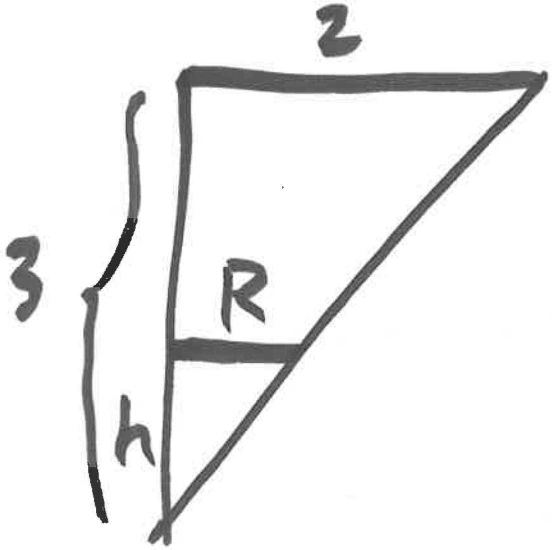
$$V'(t_0) = k \cdot 2R(t_0) \cdot R'(t_0)$$

$$= 375 \cdot 2 \cdot 0.06 \cdot 0.01 = 0.45 \text{ mm}/\text{min}^2$$

Ex 5 A cone-shaped water tank has base radius 2m, height 3m, vertex at bottom. Pump water in at a rate of $5 m^3/h$. When the volume of water is $0.5 m^3$, how fast is the water level increasing?



$$V = \frac{1}{3}\pi R^2 h$$



$$\frac{R}{h} = \frac{2}{3}$$

$$R = \frac{2}{3} h$$

$$V = \frac{1}{3}\pi \left(\frac{2}{3}h\right)^2 \cdot h$$

$$= \frac{4}{27}\pi h^3$$

$$V(t) = \frac{4}{27}\pi h(t)^3$$

$V(t_0) = 0.5$, $V'(t_0) = 5$, $h'(t_0) = ?$

$$V'(t) = \frac{4}{27}\pi \cdot 3\,h(t)^2 \cdot h'(t)$$

$$\underset{=0.5}{V(t_0)} = \frac{4}{27}\pi\, h(t_0)^3$$

$$h(t_0)^3 = \frac{0.5}{\frac{4}{27}\pi}$$

$$h(t_0) = \sqrt[3]{\frac{0.5}{\frac{4}{27}\pi}} \simeq 1.02.$$

$$V'(t_0) = \frac{4}{9}\pi\, h(t_0)^2 \cdot h'(t_0)$$

$$h'(t_0) = \frac{5}{\frac{4}{9}\pi \cdot 1.02^2} = 3.44\, m/h$$