28.1 ~ 28.2 Implicit differentiation

<u>Def</u> An <u>implicit function</u> $y(t)$ is a function defined by an equation satisfied by $y(t)$ and t.

- Sometimes you can solve for $y(t)$, get explicit function. Sometimes this is impossible.

Ex 1 $\dfrac{y(t)+1}{y(t)-1} = t$

$$y(t)+1 = t\,(y(t)-1)$$
$$= t\,y(t) - t$$

$$(1-t)\,y(t) = -t-1$$

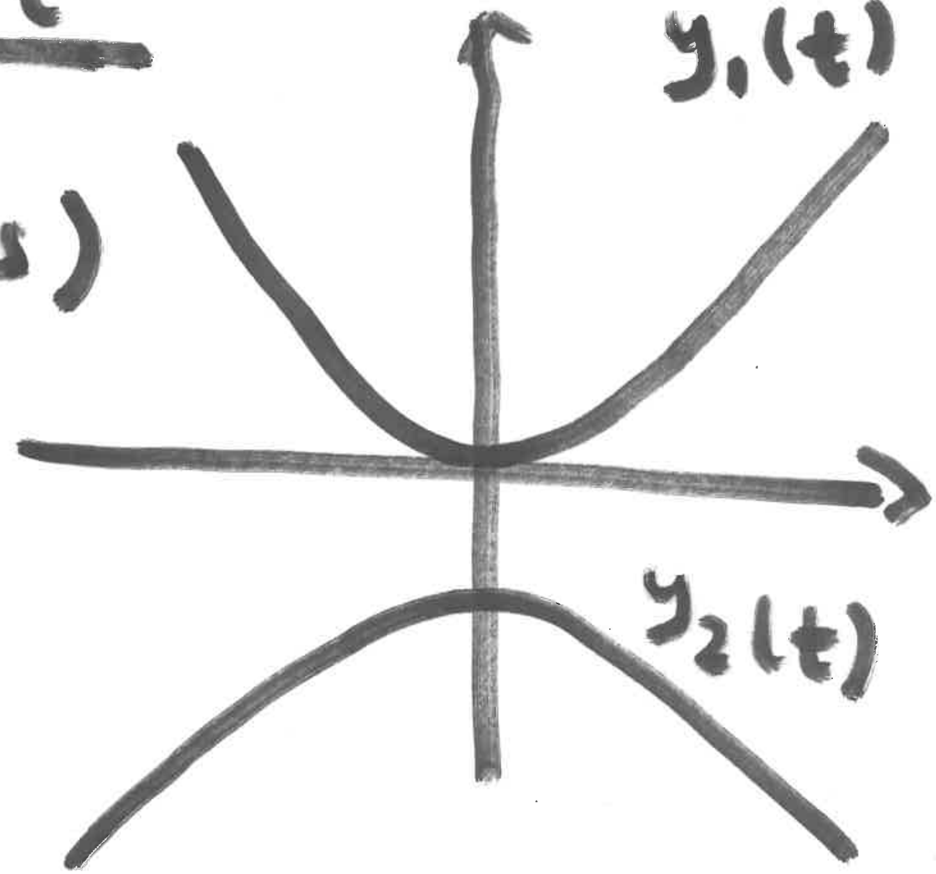
$$y(t) = \frac{-t-1}{1-t} = \frac{t+1}{t-1}$$

Ex 2 $y(t)^2 + y(t) - t^2 = 0$

$$y_1(t) = \frac{-1 + \sqrt{1+4t^2}}{2}$$

$$y_2(t) = \frac{-1 - \sqrt{1-4t^2}}{2}$$

(two possible choices)

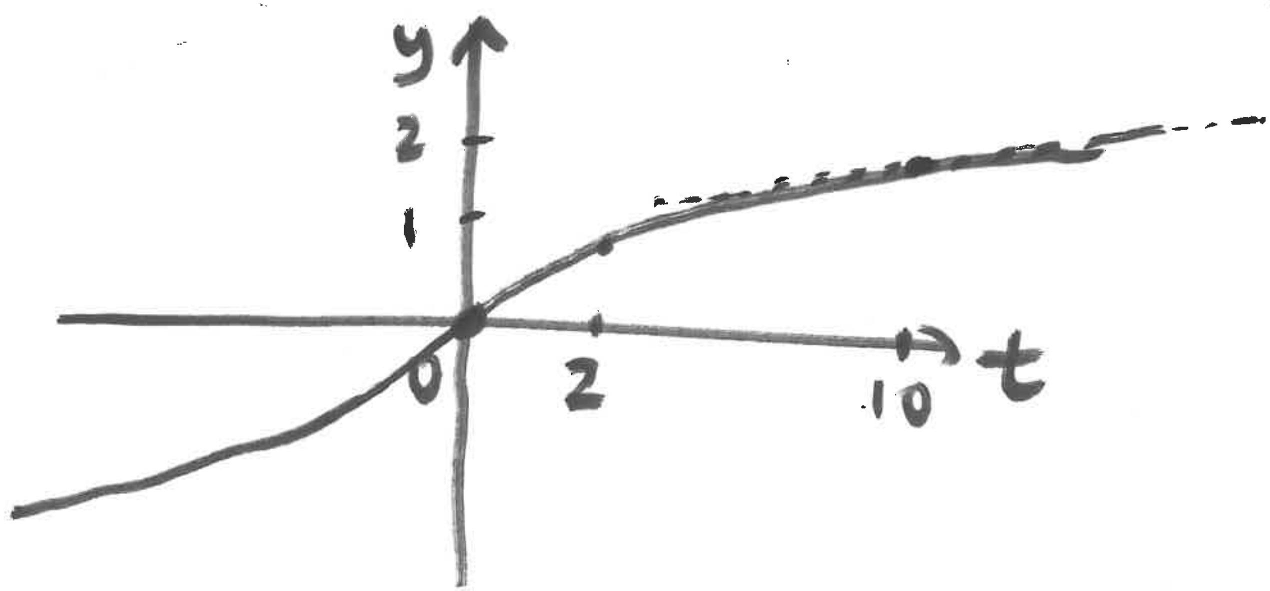


Ex 3 $y(t)^3 + y(t) = t$

$y = 0 \rightsquigarrow t = 0 \qquad y(0) = 0$

$y = 1 \rightsquigarrow t = 2 \qquad y(2) = 1$

$y = 2 \rightsquigarrow t = 10 \qquad y(10) = 2$



Ex 4 $\quad y(t)\sin y(t) = t^2 e^{2t\,y(t)}$

$t=0 \qquad y = 0, \pi, 2\pi, \ldots$

multiple choices:

$y_0(0) = 0$

$y_1(0) = \pi$

$y_2(0) = 2\pi$

$\vdots$

Implicit differentiation

to get $y'(t)$ without solving for $y(t)$.

Ex 1. Compute $y'(t)$ if $\dfrac{y(t)+1}{y(t)-1} = t$

Given $y(0) = -1$, Compute $y'(0)$.

$$\frac{y'(t)\cdot(y(t)-1)-(y(t)+1)\cdot y'(t)}{(y(t)-1)^2}=1$$

$$\frac{\cancel{y'(t)\cdot y(t)}-y'(t)-\cancel{y(t)\cdot y'(t)}-y'(t)}{(y(t)-1)^2}=1$$

$$-2\,y'(t)=(y(t)-1)^2$$

$$\cdots\cdots \quad y'(t)=-\frac{1}{2}(y(t)-1)^2$$

$$y(0)=-1 \Longrightarrow y'(0)=-\frac{1}{2}((-1)-1)^2$$

$$=-2.$$

check: $y(t) = \dfrac{t+1}{t-1}$

$$y'(t) = \frac{1\cdot(t-1) - (t+1)\cdot 1}{(t-1)^2}$$

$$= \frac{t-1-t-1}{(t-1)^2} = \frac{-2}{(t-1)^2}$$

$$-\frac{1}{2}(y(t)-1)^2 = -\frac{1}{2}\left(\frac{t+1}{t-1} - 1\right)^2$$

$$= -\frac{1}{2}\left(\frac{t+1-(t-1)}{t-1}\right)^2 = -\frac{2}{(t-1)^2}$$

<u>Ex 2</u> $\boxed{y(t)}^2 + y(t) - t^2 = 0$

Given $y(0)=0$, find $y'(0)$.

$$2y(t)\cdot y'(t) + y'(t) - 2t = 0$$

$$(2y(t)+1)\cdot y'(t) = 2t$$

$$y'(t) = \frac{2t}{2y(t)+1}$$

$$y(0)=0 \Rightarrow y'(0) = \frac{2\cdot 0}{2\cdot 0+1} = 0$$

Ex 3

$$\boxed{y}^3 + y = t$$

Given $y(10) = 2$, find $y'(10)$.

$$3 \cdot y^2 \cdot y' + y' = 1$$

$$(3y^2 + 1)\, y' = 1$$

$$y' = \frac{1}{3y^2 + 1}$$

$$y(10) = 2 \Rightarrow y'(10) = \frac{1}{3 \cdot 2^2 + 1} = \frac{1}{13}$$

Ex 4 $\quad y \sin y = t^2 e^{2ty}$

$$y' \sin y + y \cdot (\sin \boxed{y})'$$
$$= 2t e^{2ty} + t^2 (e^{\boxed{2ty}})'$$

$$y' \sin y + y \cdot \cos y \cdot y'$$
$$= 2t e^{2ty} + t^2 \cdot e^{2ty} \cdot (2 \cdot y + 2t \cdot y')$$

$$(\sin y + y \cdot \cos y - t^2 e^{2ty} 2t)\, y'$$
$$= 2t e^{2ty} + t^2 \cdot e^{2ty} \cdot 2y$$

$$y' = \frac{2t\,e^{2ty} + t^2 e^{2ty}\,2y}{\sin y + y\cdot\cos y - t^2 e^{2ty}\,2t}$$

Ex: Assume $x(t)$ and $y(t)$ satisfy $\boxed{x(t)}^2 + y(t)^2 = 100$. Find relation between $x'(t)$ and $y'(t)$. If $x(1)=6$, $x'(1)=2$, $y(1)>0$, what is $y'(1)$?

$$2\,\underbrace{x(t)}_{\checkmark}\cdot\underbrace{x'(t)}_{\checkmark}+2\,\underbrace{y(t)}_{\uparrow}\cdot\underset{?}{y'(t)}=0$$

calculate $y(1)$

$$x(1)^2+y(1)^2=100$$

$$6^2+y(1)^2=100$$

$$y(1)^2=100-36=64$$

$$y(1)=8 \quad (\text{since } y(1)>0)$$

$$2\cdot 6\cdot 2+2\cdot 8\cdot y'(1)=0$$

$$16\, y'(1) = -24$$

$$y'(1) = -\frac{24}{16} = -\frac{3}{2}$$

<u>Ex 6</u> $y(t)^2 + y(t) - t^2 = 0$

already have

$$y'(t) = \frac{2t}{2y(t)+1}$$

$$y''(t) = \frac{2\cdot(2y(t)+1) - 2t\cdot 2y'(t)}{(2y(t)+1)^2}$$

$$= \frac{4y(t) + 2 - 4t\, y'(t)}{(2y(t)+1)^2}$$

$$= \frac{4y(t) + 2 - 4t \cdot \dfrac{2t}{2y(t)+1}}{(2y(t)+1)^2}$$