# 19.4 Exponential models

Given a function $f(t)$

$$f'(t) = k f(t) \iff f(t) = Ce^{kt}$$

for some $C$.

$C$ is given by

$$C = f(0)$$

$k > 0$ : rate of growth is proportional to the amount.

exponential growth (population)

$k < 0$ : rate of decay is prop. to the amount.

exponential decay (radioactive decay)

Ex 1 Let $y(x)$ be a function satisfying $y' = -2y$ and $y(3) = 2$. Determine $y(x)$.

$$y(x) = Ce^{-2x}$$

$$y(3) = Ce^{-2\cdot 3} = 2$$

$$Ce^{-6} = 2$$

$$C = \frac{2}{e^{-6}}$$

$$C = 2\cdot e^{6}$$

$$\boxed{y(x) = 2e^{6}e^{-2x}}$$

Ex 2 let $P_1(t)$, $P_2(t)$ be the popu. of two types of bacteria. They satisfy the exp. growth models $P_1' = P_1$ and $P_2' = 4P_2$, with initial condition $P_1(0) = 16$, $P_2(0) = 1$

(1) Determine $P_1(t)$, $P_2(t)$

(2) When will their popu. be the same?

(3) At $t = 10$, which popu. is larger?

(1) $P_1(t) = 16e^t$

$P_2(t) = 1 \cdot e^{4t}$

(2) $16e^t = e^{4t}$

$16 = e^{3t}$

$\ln 16 = 3t$

$t = \frac{\ln 16}{3} \approx 0.924$

(3) $P_2$ is larger at $t = 10$.

- The <u>doubling time</u> of exp. growth model is the time it takes to double the popu.

Suppose $P(t) = Ce^{kt}$

$P(0) = C$

want $t$ such that $P(t) = 2C$

$$Ce^{kt} = 2C$$

$$e^{kt} = 2$$

$$kt = \ln 2$$

$$t = \frac{\ln 2}{k}$$

$$\boxed{\text{doubling time: } T = \frac{\ln 2}{k}}$$

- The <u>half-life</u> of exp. decay model is the time it takes to half of initial amount.

If $f'(t) = -k f(t)$  $k > 0$

then the half-life

$$\boxed{t_{1/2} = \frac{\ln 2}{k}}$$

**Ex 3** The half-life of $C^{14}$ is 5760 years. Suppose the original amount of $C^{14}$ in an organism is 10 g and its present amount is 0.002 g.

(1) What's its age?

(2) When there are only 1000 $C^{14}$ atoms left, what's its age?

(a $C^{14}$ atom has mass $2.3 \times 10^{-23}$ g)

(1) $t_{1/2} = \frac{\ln 2}{k} = 5760$

$$k = \frac{\ln 2}{5760} \approx 0.00012$$

$$f(t) = 10e^{-0.00012t}$$

$f(t) = 0.002$ , $t = ?$

$$10e^{-0.00012t} = 0.002$$

$$e^{-0.00012t} = 0.0002$$

$$-0.00012t = \ln 0.0002$$

$$t = \frac{\ln 0.0002}{-0.00012} \approx 70977 \text{ years}$$

12) $f(t) = 2.3 \times 10^{-20}$

$$10e^{-0.00012t} = 2.3 \times 10^{-20}$$

$$e^{-0.00012t} = 2.3 \times 10^{-21}$$

$$-0.00012t = \ln(2.3 \times 10^{-21})$$

$$t = \frac{\ln(2.3 \times 10^{-21})}{-0.00012} \approx 396011 \text{ years}$$

- Newton's law of cooling.

The temperature of an object satifies

$$T'(t) = -k(T(t) - T_0)$$

$k > 0$

$T_0$ : environment temperature.

Let $y(t) = T(t) - T_0$

$$y'(t) = T'(t)$$

$$\boxed{T_1 = T(0)}$$

$$y'(t) = -k\,y(t)$$

$$y(t) = C \cdot e^{-kt} \qquad C = y(0)$$

$$= T_1 - T_0$$

$$\boxed{T(t) = T_0 + (T_1 - T_0)e^{-kt}}$$

$$\lim_{t \to \infty} T(t) = T_0$$

Ex 4 Let the room temp. be 70°F. Suppose it takes 10 min for a cup of boiled water (212°F) to cool to 100°F, how long does it take to further cool to 80°F?

$$T(t) = 70 + (212 - 70)e^{-kt}$$

$$T(10) = 70 + 142e^{-10k} = 100$$

$$142e^{-10k} = 30$$

$$e^{-10k} = \frac{30}{142}$$

$$-10k = \ln\frac{30}{142}$$

$$k = \frac{\ln\frac{30}{142}}{-10} \approx 0.155$$

Want $t$ such that $T(t) = 80$

$$70 + 142e^{-0.155t} = 80$$

$$142e^{-0.155t} = 10$$

$$e^{-0.155t} = \frac{10}{142}$$

$$-0.155t = \ln\frac{10}{142}$$

$$t = \frac{\ln\frac{10}{142}}{-0.155} \approx 17.1$$

time from 100°F to 80°F is 7.1 min.

- The differential equation

$$f''(t) = -k^2 f(t).$$

$$(\sin kt)'' = (k \cos kt)'$$

$$= -k^2 \sin kt.$$

$$(\cos kt)'' = (-k \sin kt)'$$

$$= -k^2 \cos kt$$

In general,

$$f(t) = A \sin kt + B \cos kt .$$

( spring, pendulum, wave, predator-prey )