19.1 ~ 19.3 Chain rule

Def The composition of functions f and g is

$$h(x) = f(g(x))$$

denoted as $h = f \circ g$.

- f is called outer func., g inner func.
- In general, $f \circ g \neq g \circ f$

Ex 1 Compute $f \circ g$ and $g \circ f$

① $f(\boxed{x}) = e^{\boxed{x}}$, $g(\boxed{x}) = 3\boxed{x}^2$

$(f \circ g)(x) = f(g(x))$

$= f(\boxed{3x^2}) = e^{3x^2}$

$(g \circ f)(x) = g(f(x))$

$= g(\boxed{e^x}) = 3(e^x)^2 = 3e^{2x}$

② $f(x) = \cos x$, $g(x) = \frac{1}{x}$

$(f \circ g)(x) = f(g(x))$

$= f\left(\frac{1}{x}\right) = \cos\frac{1}{x}$

$(g \circ f)(x) = g(f(x))$

$= g(\cos x) = \frac{1}{\cos x}$

③ $f(x) = g(x) = x^2+1$

$(f\circ f)(x) = f(f(x))$

$= f(x^2+1) = (x^2+1)^2 + 1 = x^4+2x^2+2$

④ $f(x) = g(x) = \dfrac{1}{x^2-1}$

$(f\circ f)(x) = f\left(\dfrac{1}{x^2-1}\right) = \dfrac{1}{\left(\dfrac{1}{x^2-1}\right)^2 - 1}$

$= \dfrac{(x^2-1)^2}{1-(x^2-1)^2}$

- The chain rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Notation: $f'(x)$ also denoted $\dfrac{df}{dx}$

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

- Usage:

① first, box the inner function g

② take derivative as if g is the variable

③ then multiply by g'

Ex 2 Compute

① $\left(\left(\boxed{x^2+1}\right)^{10} \right)'$

$= 10 \cdot \boxed{x^2+1}^{9} \cdot (x^2+1)'$

$= 10 \cdot (x^2+1)^9 \cdot 2x = 20x(x^2+1)^9$

$f(x) = x^{10}, \quad g(x) = x^2+1$

$(f \circ g)(x) = f(x^2+1) = (x^2+1)^{10}$

② $\left(e^{\boxed{x^3+x}}\right)'$

$= e^{\boxed{x^3+x}} \cdot (x^3+x)'$

$= e^{x^3+x} \cdot (3x^2+1)$

③ $\left(\sin \boxed{\frac{1}{x}}\right)'$

$= \cos \boxed{\frac{1}{x}} \cdot \left(\frac{1}{x}\right)' = \cos\frac{1}{x} \cdot (-1) \cdot \frac{1}{x^2}$

④ $\left(\cos\left(\boxed{e^{2x}}\right)\right)' = -\sin\left(e^{2x}\right)\cdot\left(e^{2x}\right)'$

$= -\sin\left(e^{2x}\right)\cdot 2e^{2x}$

⑤ $\left(\underline{x}\,\underline{e^{x^2}}\right)' = (x)'\cdot e^{x^2} + x\cdot\left(e^{x^2}\right)'$

$= e^{x^2} + x\cdot e^{x^2}\cdot 2x$

$= e^{x^2}\left(1+2x^2\right)$

⑥ $$\left(\frac{x}{1+\cos(e^{-x})}\right)'$$

$$= \frac{(x)'\cdot(1+\cos(e^{-x})) - x\cdot(1+\cos(e^{-x}))'}{(1+\cos(e^{-x}))^2}$$

$$= \frac{1+\cos(e^{-x}) - x\cdot(-\sin(e^{-x}))\cdot(-e^{-x})}{(1+\cos(e^{-x}))^2}$$

$$= \frac{1+\cos(e^{-x}) - xe^{-x}\sin(e^{-x})}{(1+\cos(e^{-x}))^2}$$

• Several derivations:

(assume we know $(e^x)' = e^x$)

① $(e^{kx})' = e^{kx} \cdot (kx)' = ke^{kx}$

② Let $f(x) = \ln x$

$$f(e^x) = \ln(e^x) = x$$

$$f'(e^x) \cdot (e^x)' = 1$$

$$f'(e^x) = \frac{1}{e^x} \Rightarrow f'(x) = \frac{1}{x}$$

③ $(x^n)' = \left(e^{\ln(x^n)}\right)'$

$= \left(e^{n \ln x}\right)'$

$= e^{n \ln x} \cdot (n \ln x)'$

$= x^n \cdot n \cdot \frac{1}{x} = n x^{n-1}$

- Sometimes you need chain rule multiple times.

Ex 3 Compute

① $\left(e^{\boxed{\sin(2x-1)}}\right)'$

$= e^{\sin(2x-1)} \cdot \left(\sin(\boxed{2x-1})\right)'$

$= e^{\sin(2x-1)} \cdot \cos(2x-1) \cdot 2$.

② $\left(\sqrt{\boxed{1+\sin^2 x}}\right)'$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{1+\sin^2 x}} \cdot \left(1 + \left(\boxed{\sin x}\right)^2\right)'$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{1+\sin^2 x}} \, 2\sin x \cdot \cos x$$

$$= \frac{\sin x \cos x}{\sqrt{1+\sin^2 x}}$$

19.5 Higher order derivatives

Def The <u>second order derivative</u> of f is

$$f''(x) = \left(f'(x)\right)'$$

(also denoted $\frac{d^2 f}{dx^2}$)

Similarly define $f''', f^{(4)}, \ldots, f^{(n)}$

<u>Ex 4</u> Compute

① $(e^{-3x})'' = (-3e^{-3x})'$

$= -3\cdot(-3)e^{-3x} = 9e^{-3x}$

② $(\sin(2x))'' = (2\cos(2x))'$

$= 2\cdot 2\cdot(-\sin(2x)) = -4\sin 2x$

③ $(x^4+x^2-1)'''$

$= (4x^3+2x)''$

$= (4\cdot 3x^2+2)'$

$= 4\cdot 3\cdot 2x = 24x$

$(x^4+x^2-1)^{(100)} = 0$

④ $\left(\sqrt{\boxed{1+x^2}}\right)''$

$$=\left(\frac{1}{2}\cdot\frac{1}{\sqrt{1+x^2}}\cdot 2x\right)'$$

$$=\left(\frac{x}{\sqrt{1+x^2}}\right)'$$

$$=\frac{(x)'\sqrt{1+x^2}-x\cdot\left(\sqrt{1+x^2}\right)'}{1+x^2}$$

$$=\frac{\sqrt{1+x^2}-x\cdot\frac{x}{\sqrt{1+x^2}}}{1+x^2}$$

$$= \frac{1+x^2 - x^2}{(1+x^2)\cdot\sqrt{1+x^2}}$$

$$= \frac{1}{(1+x^2)^{3/2}}$$