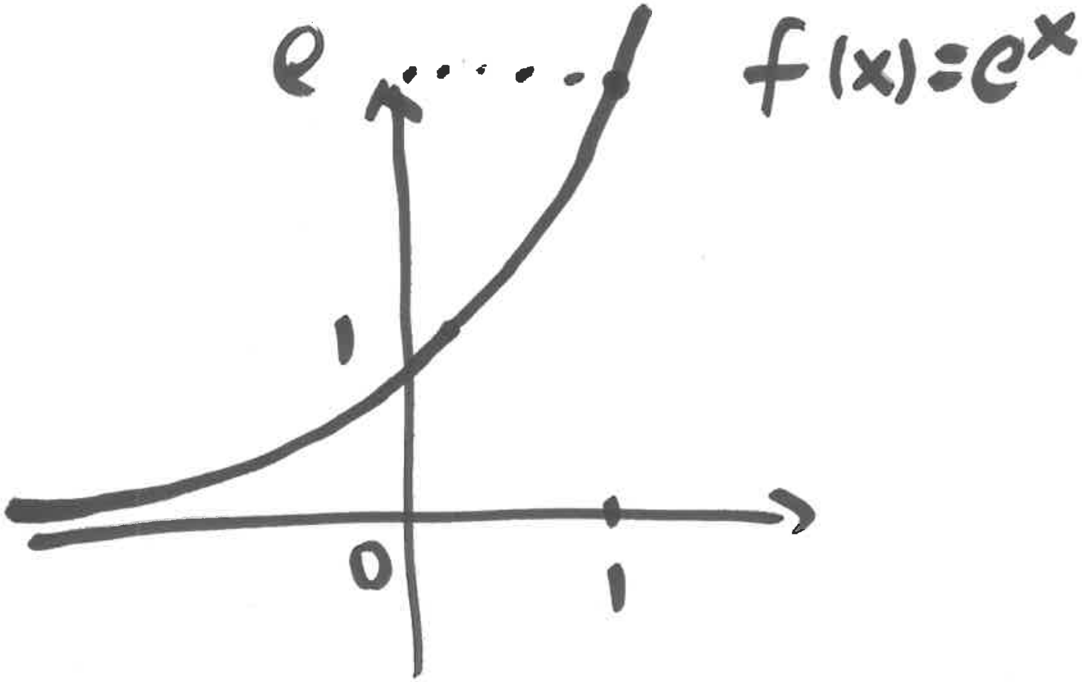
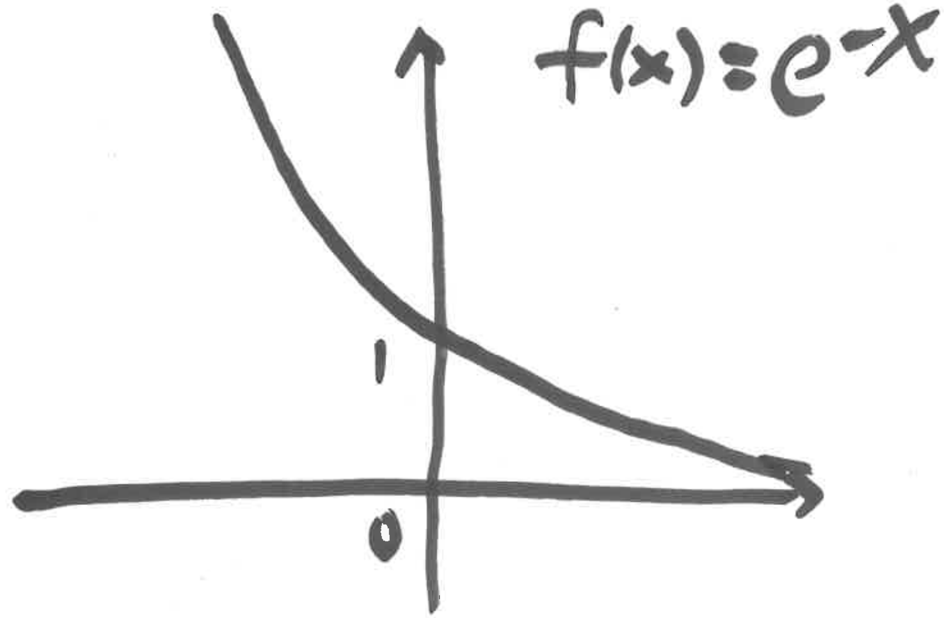
18.3 ~ 18.6 More derivative formulas

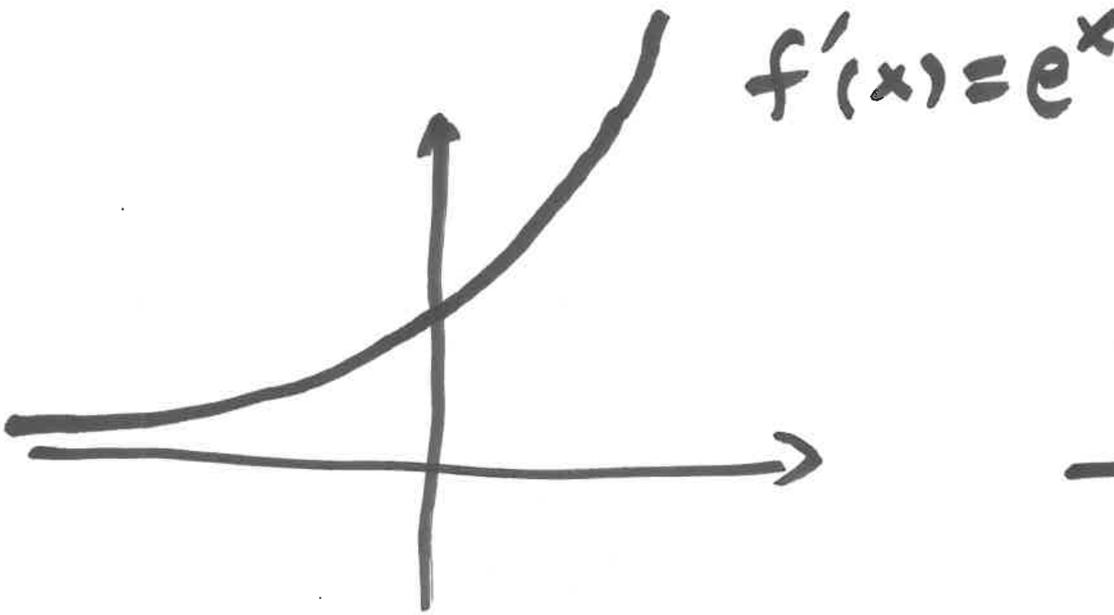
- $(e^{kx})' = k e^{kx}$

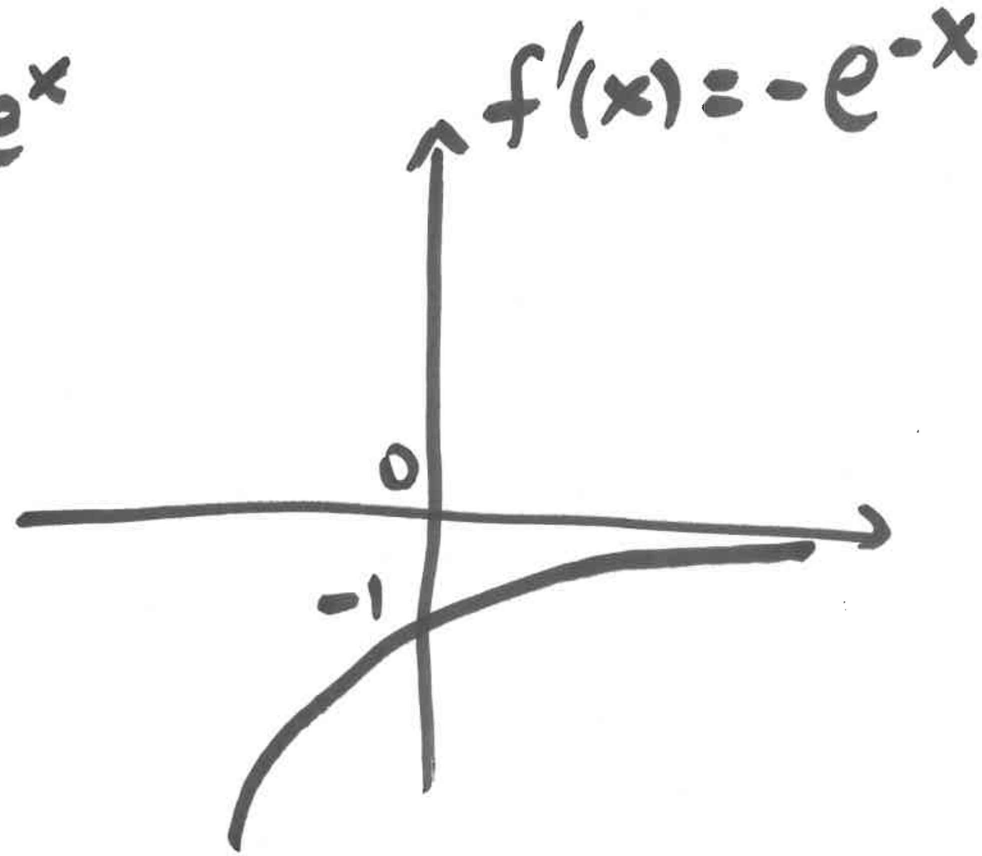
$(a^x)' = (\ln a) \cdot a^x \qquad a > 0$

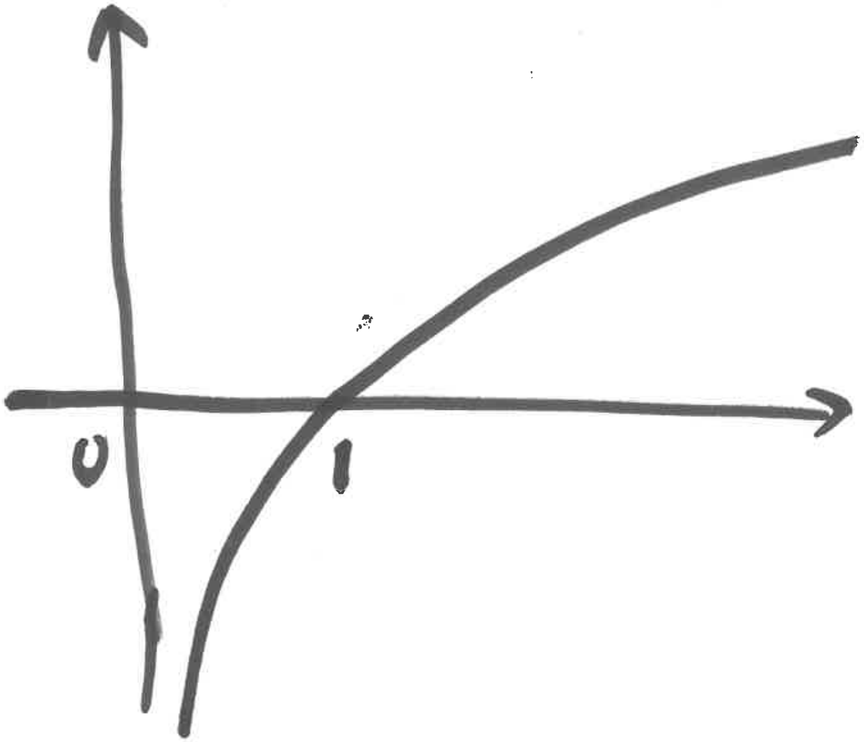
$(\ln x)' = \dfrac{1}{x}$


$f(x)=e^x$
e
1
0
1

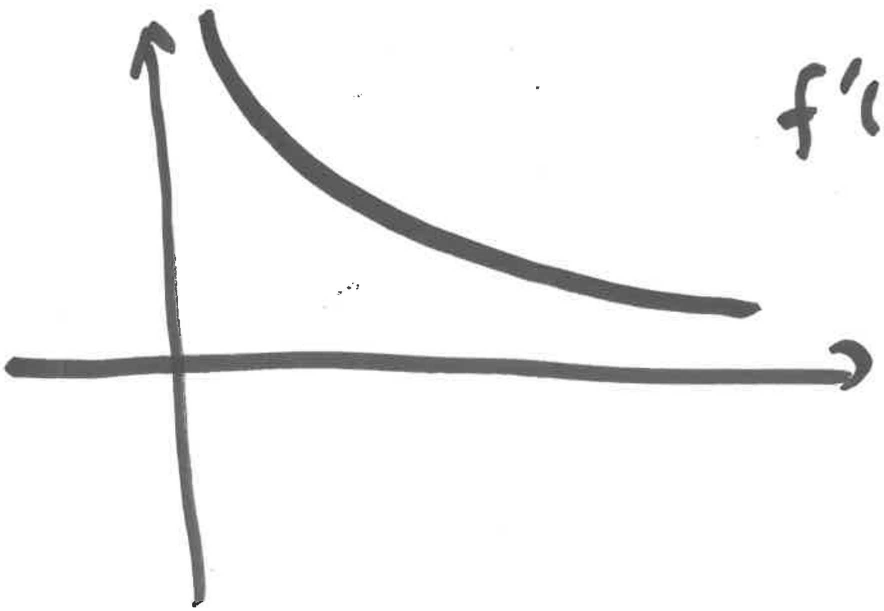

$f(x)=e^{-x}$
1
0


$f'(x)=e^x$


$f'(x)=-e^{-x}$
0
-1



$f(x) = \ln x$



$f'(x) = \frac{1}{x}$

Ex 1 Compute

① $\left(\underbrace{(x^2-1)}\underbrace{e^{-x}}\right)'$

$= 2x \cdot e^{-x} + (x^2-1)\cdot(-1)e^{-x}$

$= (2x - x^2 + 1)e^{-x}$

② $(1 + 2\ln x)'$

$= 2 \cdot \frac{1}{x}$

③ $\left((\ln x)^2 \right)'$

$= \left(\underbrace{(\ln x)} \cdot \underbrace{(\ln x)} \right)'$

$= \frac{1}{x} \cdot \ln x + \ln x \cdot \frac{1}{x}$

$= \frac{2 \ln x}{x}$

Ex 2 Let the population of a bacteria $P(t)$ given by an exponential growth model (million, day)

$$P(t) = Ae^{kt} \qquad A>0,\ k>0$$

Suppose at $t=0$ the pop. is 2M. and at $t=3$ the pop. is doubled.

(1) Find A, k.

$$P(0) = A \cdot e^{k \cdot 0} = \underline{A = 2}$$

$$P(3) = A \cdot e^{3k} = 4$$

$$2e^{3k} = 4$$

$$e^{3k} = 2$$

$$3k = \ln 2$$

$$\underline{k = \frac{1}{3} \ln 2}$$

(2) What is the average growth rate from day 2 to day 4?

$$\frac{\Delta P}{\Delta t} = \frac{P(4) - P(2)}{4 - 2}$$

$$= \frac{2e^{\frac{1}{3}\ln 2 \cdot 4} - 2e^{\frac{1}{3}\ln 2 \cdot 2}}{2}$$

$$\approx 0.932 \ \text{M/day}.$$

(3) What is the instantaneous growth rate at day 3?

$$P(t) = A e^{kt}$$

$$P'(t) = A \cdot k e^{kt}$$

$$= 2 \cdot \frac{1}{3} \ln 2 \cdot e^{\frac{1}{3} \ln 2 \cdot t}$$

$$P'(3) = \frac{2}{3} \ln 2 \cdot e^{\ln 2} = \frac{4}{3} \ln 2$$

$$\approx 0.924$$

(4) When will the growth rate reach 100 M/day ?

$$p'(t) = 100$$

$$\frac{2}{3}\ln 2 \cdot e^{\frac{1}{3}\ln 2 \cdot t} = 100$$

$$e^{\frac{1}{3}\ln 2 \cdot t} = \frac{100}{\frac{2}{3}\ln 2} = \frac{150}{\ln 2}$$

$$\frac{1}{3}\ln 2 \cdot t = \ln\left(\frac{150}{\ln 2}\right)$$

$$t = \frac{3\ln\left(\frac{150}{\ln 2}\right)}{\ln 2}$$

$$\approx 23$$

$$P(t) = Ae^{kt}$$

$$P'(t) = Ake^{kt}$$

$$\underline{P'(t) = k\,P(t)}$$

- Quotient rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

in particular, when $f = 1$

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

Ex 3 Compute

① $\left(\dfrac{x^2-1}{x^2+1}\right)'$ $\qquad f = x^2-1$

$\qquad g = x^2+1$

$$= \frac{\overset{f'}{2x}\cdot(\overset{g}{x^2+1}) - (\overset{f}{x^2-1})\cdot \overset{g'}{2x}}{\underbrace{(x^2+1)^2}_{g^2}}$$

$$= \frac{\cancel{2x^3}+2x-\cancel{2x^3}+2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

② $\left(\frac{1}{\ln x}\right)' = -\frac{\frac{1}{x}}{(\ln x)^2} = -\frac{1}{x(\ln x)^2}$

$g = \ln x$

③ $\left(\frac{e^{2x}}{e^{-x}}\right)'$

$= \frac{2e^{2x}\cdot e^{-x} - e^{2x}(-1)\cdot e^{-x}}{(e^{-x})^2}$

$= \frac{2e^{x} + e^{x}}{e^{-2x}} = \frac{3e^{x}}{e^{-2x}} = 3e^{3x}$

④ $\left(\frac{\ln x}{x^2}\right)'$

$$= \frac{\frac{1}{x}\cdot x^2 - \ln x \cdot 2x}{(x^2)^2}$$

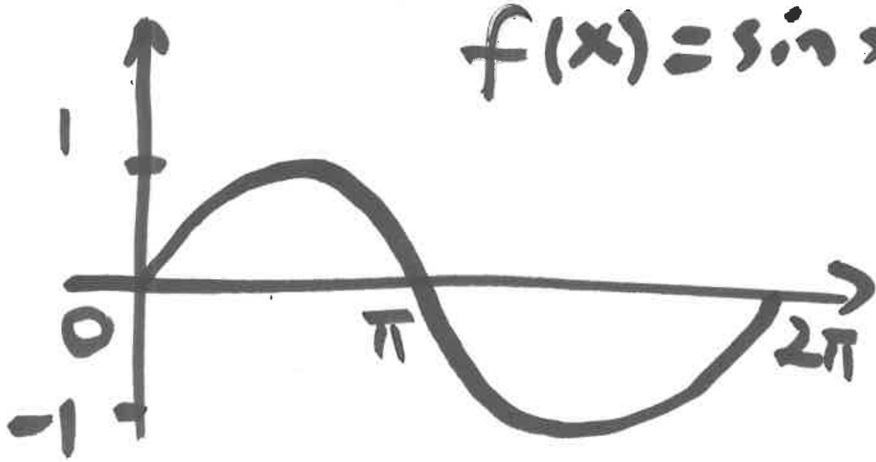
$$= \frac{x - \ln x \cdot 2x}{x^4}$$

$$= \frac{1 - 2\ln x}{x^3}$$

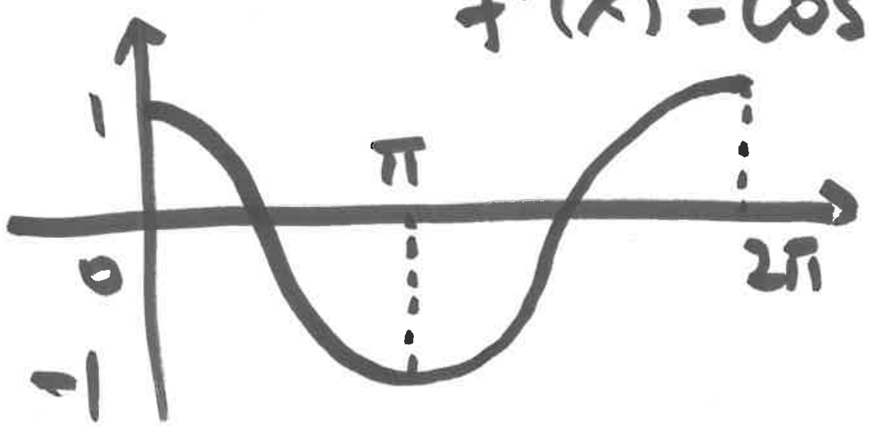
- $(\sin Cx)' = C \cdot \cos Cx$

  $(\cos Cx)' = -C \sin Cx$

$f(x) = \sin x$



$f'(x) = \cos x$



Ex 4 Compute

① $(\sin x - \cos 2x)'$

$= \cos x - 2\cdot(-\sin(2x))$

$= \cos x + 2\sin 2x$

② $(\underbrace{\sin x}\ \underbrace{\cos x})'$

$= \cos x \cdot \cos x + \sin x \cdot (-\sin x)$

$= \cos^2 x - \sin^2 x$

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$(\sin x\cos x)' = \left(\frac{1}{2}\sin 2x\right)'$$

$$= \frac{1}{2}2\cos 2x = \cos 2x$$

③ $(\tan x)' = \left(\dfrac{\sin x}{\cos x}\right)'$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$ (remember)

④ $\left(\dfrac{\sin x}{x}\right)' = \dfrac{\cos x \cdot x - \sin x}{x^2}$

Def $f(x)$ is <u>differentiable</u> at $x=a$

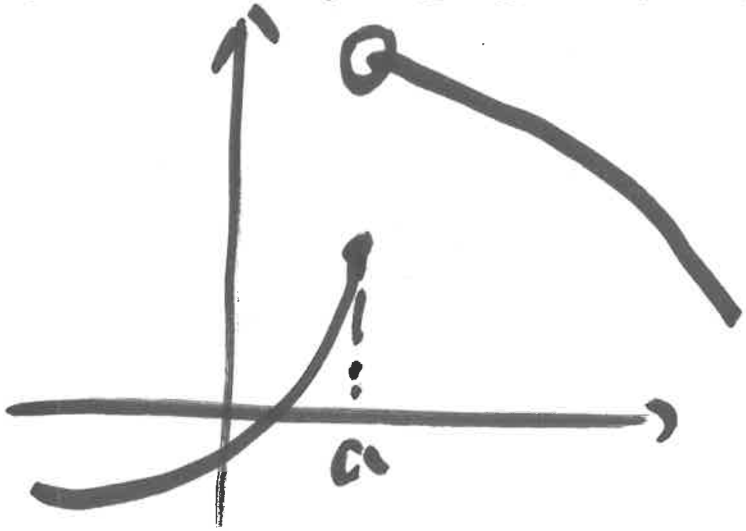
if $\displaystyle\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists.

↑

def. of $f'(a)$

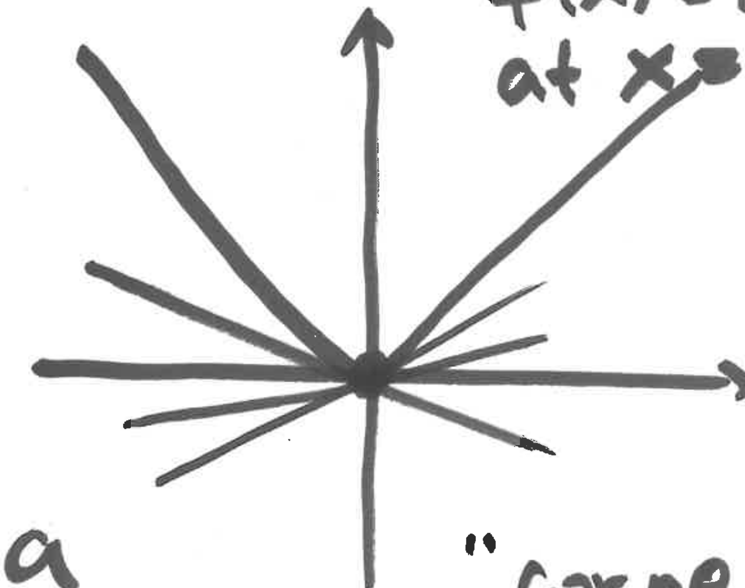
- If $f(x)$ is differentiable at $x=a$, then $f(x)$ is continuous at $x=a$.

non-differentiable functions

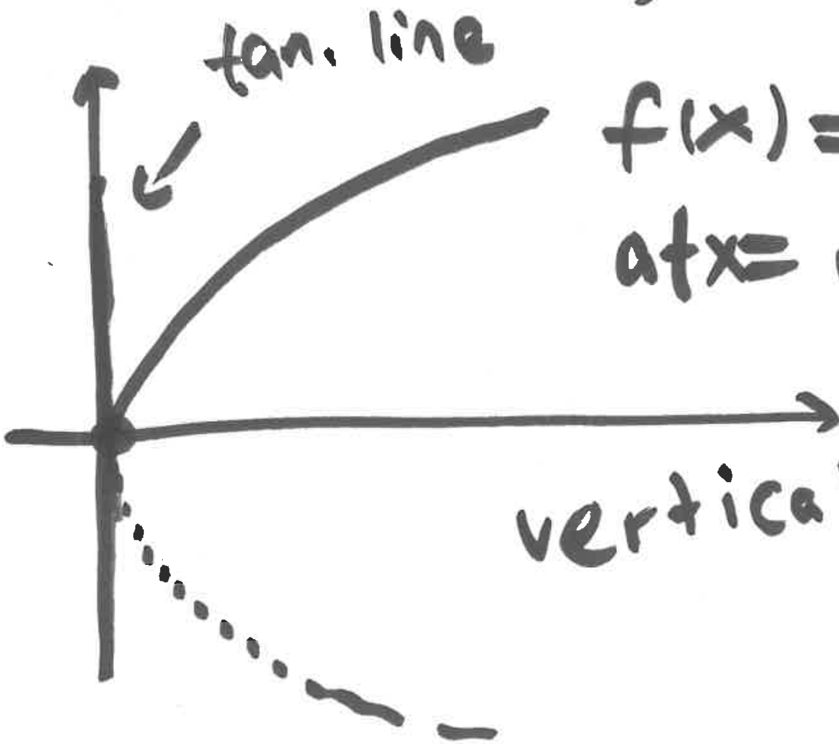


discontinuity at $x=a$

$f(x)=|x|$
at $x=0$



"corner"



$f(x)=\sqrt{x}$
at $x=0$

$f'(x)=\dfrac{1}{2\sqrt{x}}$
not defined at
$x=0$.

vertical tangent line