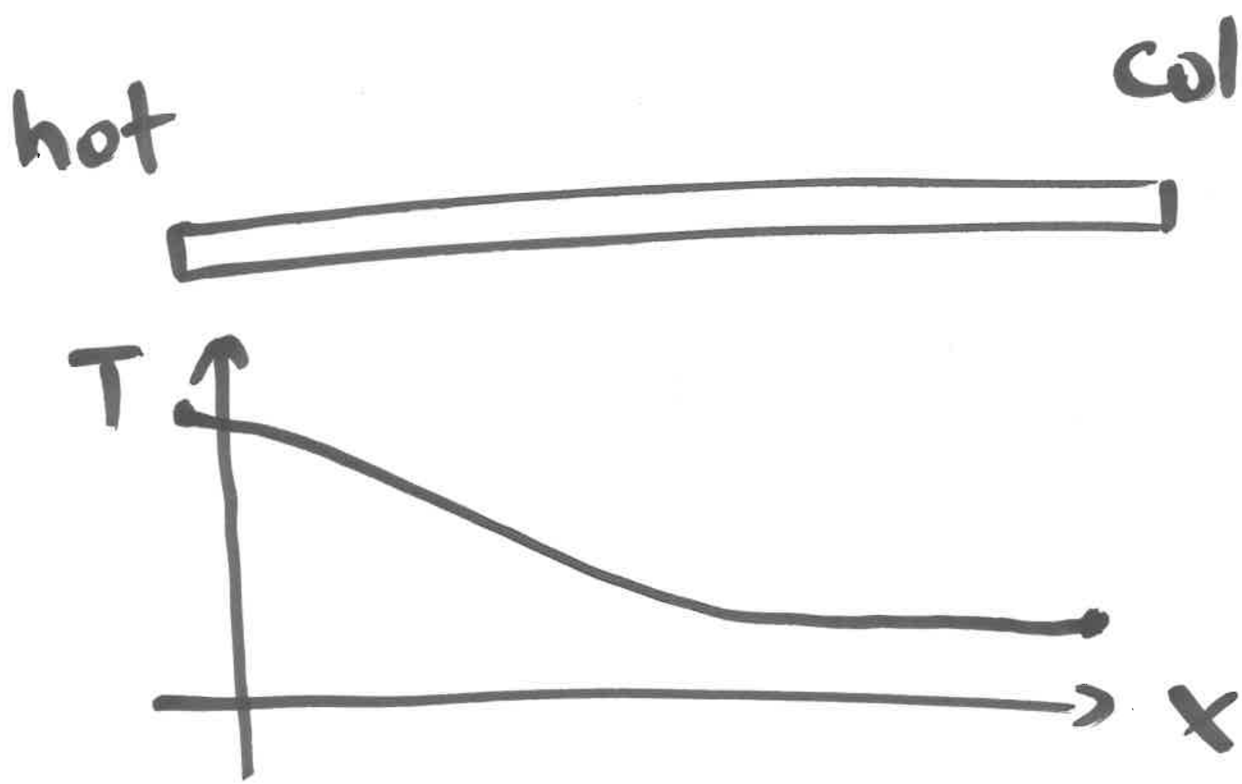
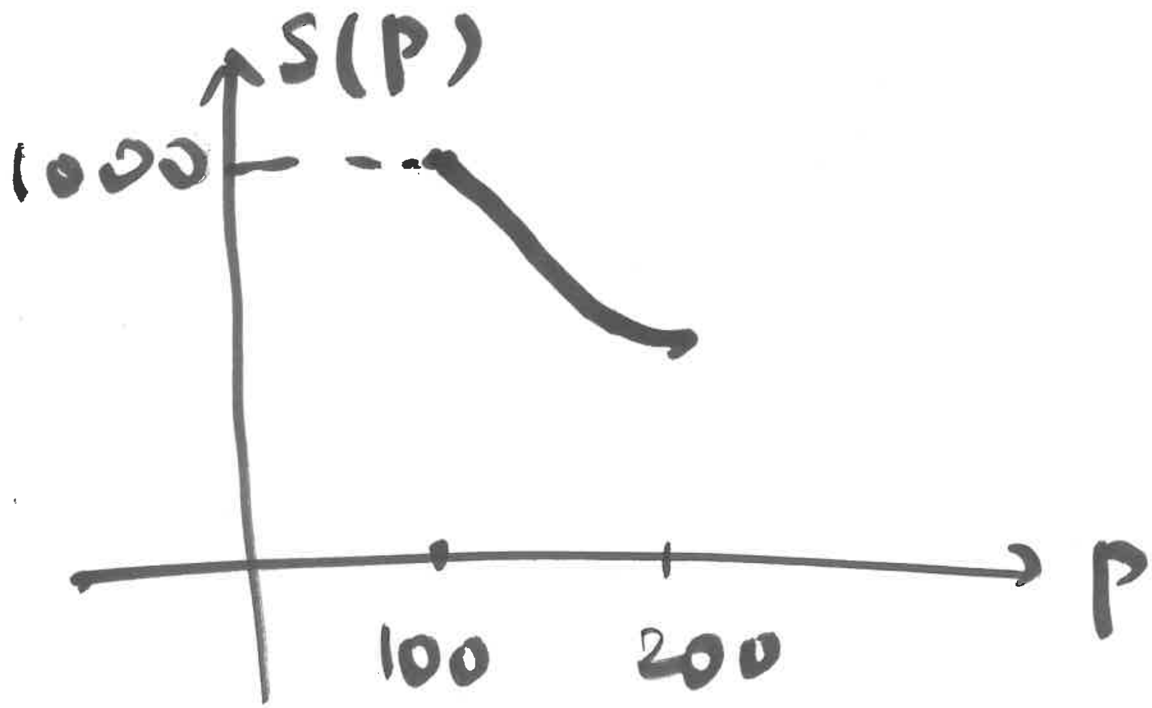
18.1, 18.2 Derivative of functions

- Temperature distribution on a rod



- Price increases → sale amount decreases



<u>Def</u> The <u>derivative</u> of $f(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- last $f'(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a}$

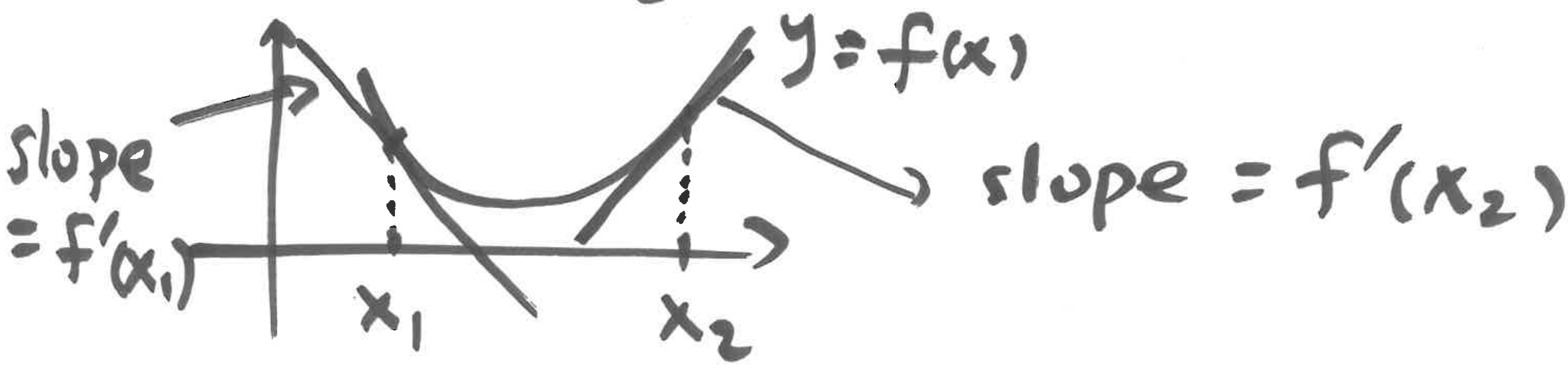
$a \rightsquigarrow x \qquad t \rightsquigarrow x+h$

- $f'(x)$ is a function

- Meaning of $f'(x)$:

(1) instantaneous rate of change of f at x

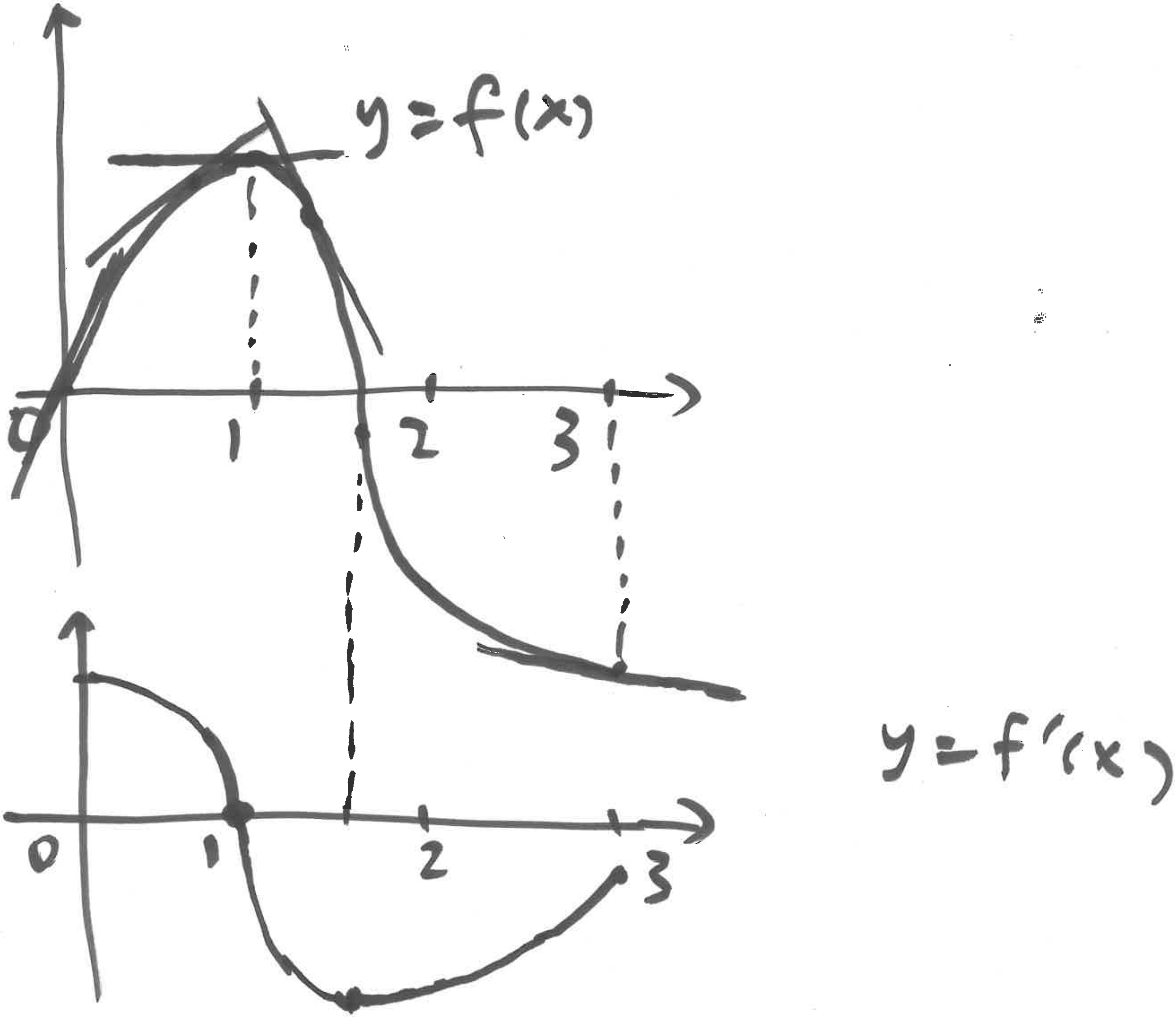
(2) slope of tangent line to the graph $(x, f(x))$



- $f'(x) > 0 \Rightarrow f$ is increasing (near x)

- $f'(x) < 0 \Rightarrow f$ is decreasing

Ex 1 Given graph of $f(x)$, sketch graph of $f'(x)$.


y = f(x)
0
1
2
3
y = f'(x)

Ex 3 Compute $f'(x)$
(use definition)

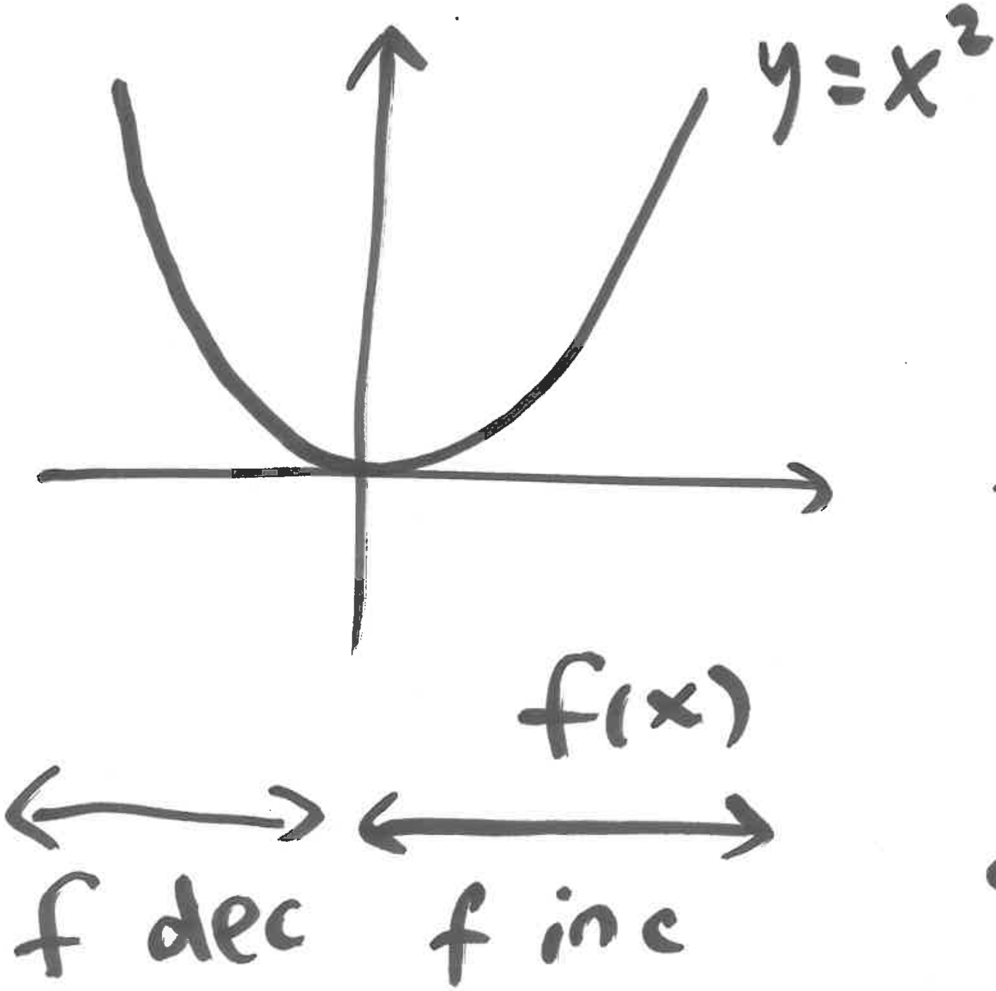
① $f(x) = x^2$

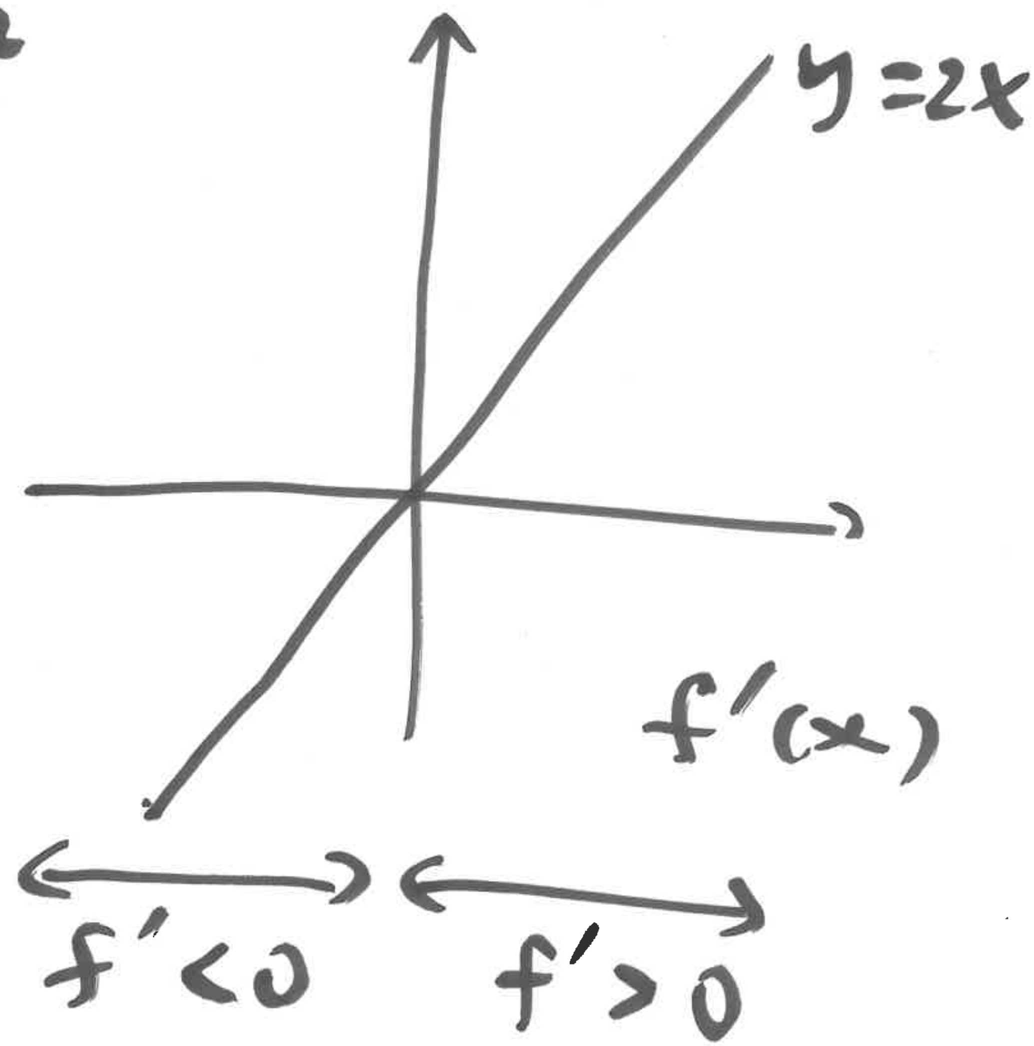
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{- x^2}}{h}$$

$$= \lim_{h \to 0} (2x + h) = 2x$$





② $f(x) = \sqrt{x}$

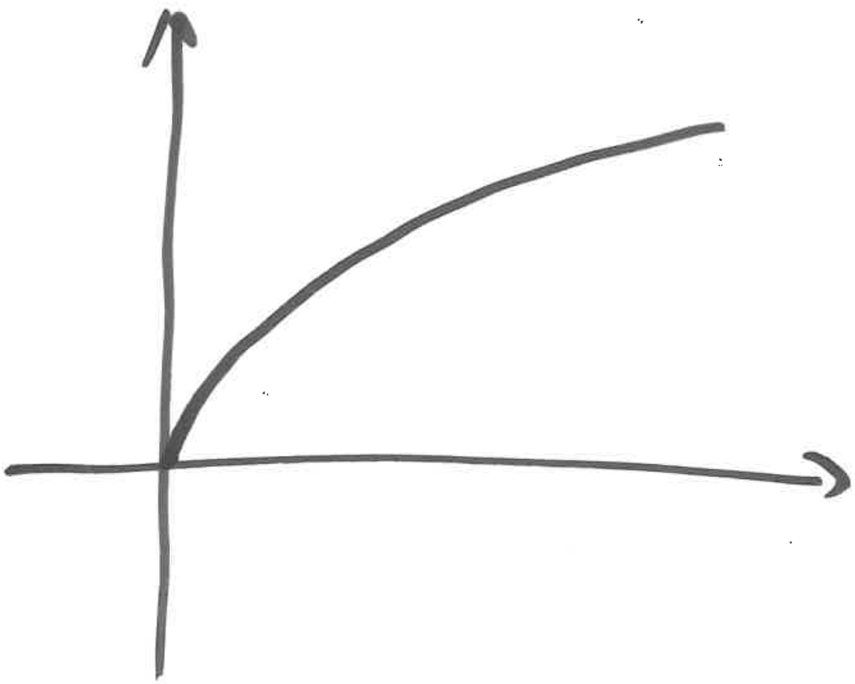
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

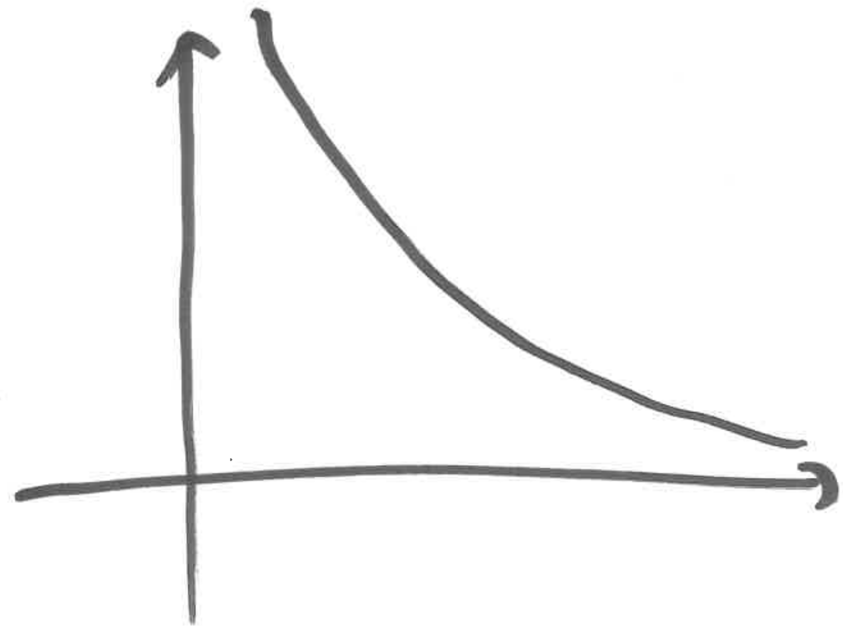
$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h \cdot (\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$



$y = \sqrt{x}$



$y = \frac{1}{2\sqrt{x}}$

$f(x)$ inc. $\qquad f'(x) > 0$

$(x^2)' = 2x$ $(n=2)$, $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$ $(n=\frac{1}{2})$

- $(x^n)' = n x^{n-1}$

(for any real number n)

$n=3$: $(x^3)' = 3x^2$

$n=0$: $(1)' = 0$

$(c)' = 0$

constant

$n = -1$: $\left(\frac{1}{x}\right)' = (-1) \cdot x^{-2}$

$$= -\frac{1}{x^2}$$

$n = -2$: $\left(\frac{1}{x^2}\right)' = (-2)\, x^{-3}$

$$= -\frac{2}{x^3}$$

- Properties:

$$(f+g)' = f' + g'$$

$$(cf)' = c \cdot f'$$

- $(e^{kx})' = k e^{kx}$

$$(e^{x})' = e^{x} \qquad (e^{-2x})' = -2e^{-2x}$$

Ex 3 Compute

(a) $(x^2 - 1 - e^{2x})'$

$= (x^2)' + (-1)' + (-e^{2x})'$

$= 2 \cdot x^1 + 0 + (-1) \cdot 2e^{2x}$

$= 2x - 2e^{2x}$

(2) $\left(\frac{2}{x^3} + 3e^{-x} \right)'$

$$= \left(2 \cdot x^{-3} + 3 \cdot e^{(-1) \cdot x} \right)'$$

$$= 2 \cdot (-3) x^{-4} + 3 \cdot (-1) e^{(-1)x}$$

$$= -\frac{6}{x^4} - 3e^{-x}$$

③ $\left((x^3+1)^2 \right)'$

$= \left(x^6 + 2x^3 + 1 \right)'$

$= 6x^5 + 2 \cdot 3x^2 + 0$

$= 6x^5 + 6x^2$

- Product rule

$$(f \cdot g)' = f'g + fg'$$

when $g(x) = c$

$$(c \cdot f)' = f' \cdot c \cancel{+ f \cdot c'}$$

Ex 4 Compute

① $$(\underbrace{x^2}_{f}\,\underbrace{e^x}_{g})'$$

$$= (x^2)'e^x + x^2(e^x)'$$

$$= 2xe^x + x^2e^x$$

$$= (2x + x^2)e^x.$$

② $\left(\left(3x - \frac{1}{x}\right) e^{-2x} \right)'$

$$= \left(3x - \frac{1}{x}\right)' \cdot e^{-2x} + \left(3x - \frac{1}{x}\right) \cdot \left(e^{-2x}\right)'$$

$$= \left(3 + \frac{1}{x^2}\right) \cdot e^{-2x} + \left(3x - \frac{1}{x}\right) \cdot (-2) e^{-2x}$$

③ $\left(e^{-3x} \cdot e^{3x}\right)'$

$$= \left(e^{-3x+3x}\right)' = (1)' = 0$$

• $(a^x)' = (\ln a)\, a^x$

$$a^x = e^{\ln(a^x)} = e^{x \cdot \underbrace{\ln a}_{\text{"}k\text{"}}}$$

$$(a^x)' = (\ln a) \cdot e^{x \cdot \ln a}$$

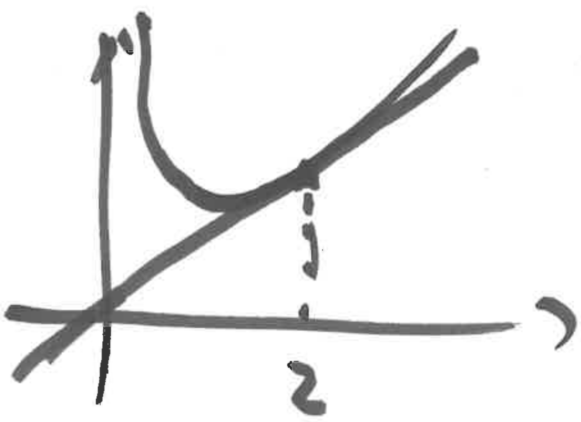
<u>Ex</u> 5 $\quad (2^x \cdot x)' = (2^x)' \cdot x + 2^x \cdot (x)'$

$$= (\ln 2) \cdot 2^x \cdot x + 2^x \cdot 1$$

Ex 6 Compute the tangent line to the graph of $y = x + \frac{1}{x}$ at $\left(2, \frac{5}{2}\right)$

$f(x) = x + \frac{1}{x}$

This line goes through $\left(2, \frac{5}{2}\right)$ with slope $f'(2)$



$$f'(x) = 1 - \frac{1}{x^2}$$

slope

$$f'(2) = 1 - \frac{1}{2^2} = \frac{3}{4}$$

$$\underline{y - \frac{5}{2} = \frac{3}{4}(x-2)}$$