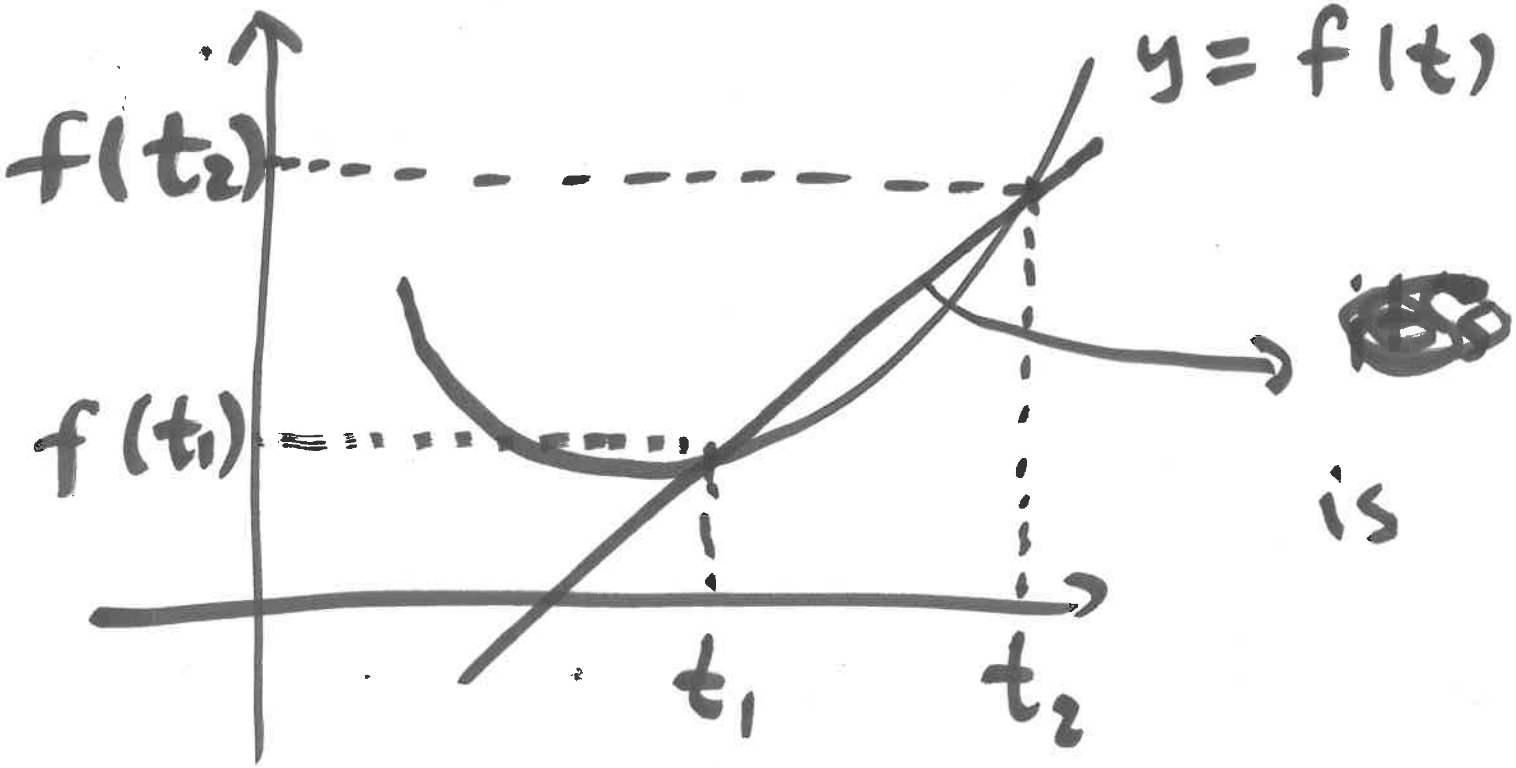
17.1 ~ 17.6 Rate of change, derivative

<u>Def</u> Given a function $f(t)$, the <u>average rate of change</u> over the time interval $[t_1, t_2]$ is defined as

$$\frac{\Delta f}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

change in f

change in t.



its slope is $\dfrac{\Delta f}{\Delta t}$.

- Describe how fast a quantity is changing. $\dfrac{\Delta f}{\Delta t} > 0$: f is increasing

$\dfrac{\Delta f}{\Delta t} < 0$: f is decreasing.

Ex 1 Population of USA

t / year	$P(t)$ / million
1940	132.1
1960	180.7
1980	226.5
2000	282.2

Compute average rate of change of $P(t)$ in $[1940, 2000]$ and $[1960, 1980]$.

For [1940, 2000],

$$\frac{\Delta P}{\Delta t} = \frac{P(2000) - P(1940)}{2000 - 1940}$$

$$= \frac{282.2 - 132.1}{60} \approx 2.50 \text{ million/year}$$

For [1960, 1980]

$$\frac{\Delta P}{\Delta t} = \frac{P(1980) - P(1960)}{1980 - 1960}$$

$$= \frac{226.5 - 180.7}{20} = 2.29 \text{ million/year}.$$

__Ex 2__ The height of a falling object $h(t) = 100 - 5t^2$

meter (h), second (t)

Compute average rate of change in $[1, 2]$, $[1, 1.1]$, $[1, 1.01]$.

For $[1, 2]$, $\dfrac{\Delta h}{\Delta t} = \dfrac{h(2) - h(1)}{2 - 1}$

$$= \frac{(100 - 5 \cdot 2^2) - (100 - 5 \cdot 1^2)}{1} = -15$$

For $[1, 1.1]$, $\frac{\Delta h}{\Delta t} = \frac{h(1.1) - h(1)}{1.1 - 1}$

$$= \frac{(100 - 5 \cdot 1.1^2) - (100 - 5 \cdot 1^2)}{1.1 - 1} = -10.5$$

For $[1, 1.01]$,

$$\frac{\Delta h}{\Delta t} = \frac{(100 - 5 \cdot 1.01^2) - (100 - 5 \cdot 1^2)}{1.01 - 1} = -10.05$$

Ex 3 Let $f(t) = t^3$.

Compute average rate of change in $[0, 2]$, $[0.9, 1.1]$, $[0.99, 1.01]$

For $[0, 2]$,

$$\frac{\Delta f}{\Delta t} = \frac{2^3 - 0^3}{2 - 0} = 4$$

For $[0.9, 1.1]$

$$\frac{\Delta f}{\Delta t} = \frac{1.1^3 - 0.9^3}{1.1 - 0.9} = 3.01$$

For $[0.99, 1.01]$

$$\frac{\Delta f}{\Delta t} = \frac{1.01^3 - 0.99^3}{1.01 - 0.99} = 3.0001$$

$$\longrightarrow \underline{3}$$

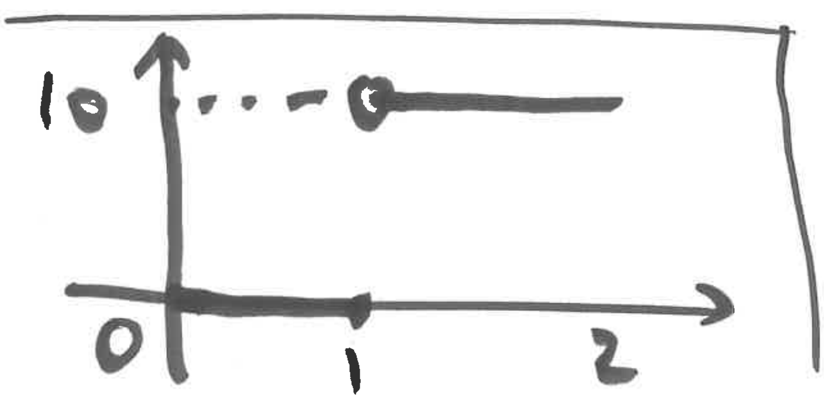
<u>Ex 4</u> Let $f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ 10 & 1 < t \le 2. \end{cases}$

Compute rate of change in

$[0,2]$, $[0.9, 1.1]$

For $[0,2]$, $\dfrac{\Delta f}{\Delta t} = \dfrac{10-0}{2-0} = 5$

For $[0.9, 1.1]$, $\dfrac{\Delta f}{\Delta t} = \dfrac{10-0}{1.1-0.9} = 50$

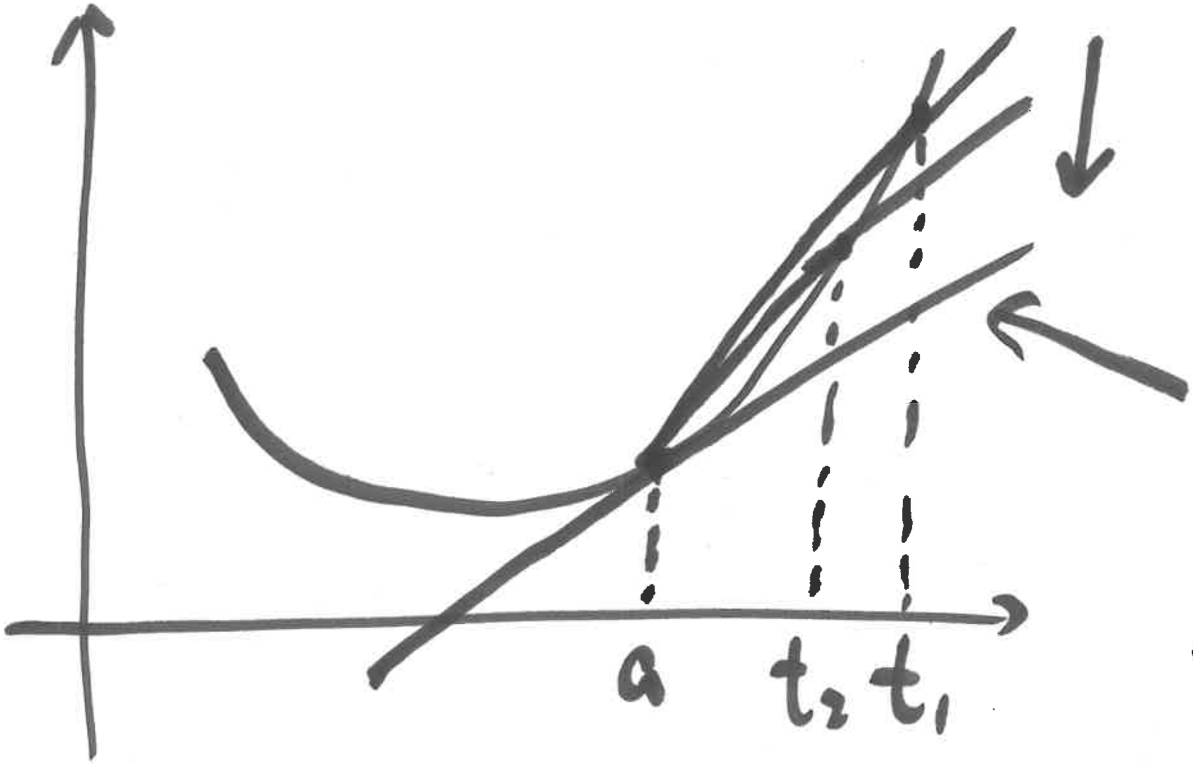


$\longrightarrow \infty$

Def The <u>derivative</u> of $f(t)$ at $t=a$ is

$$f'(a) = \lim_{t \to a} \underbrace{\frac{f(t)-f(a)}{t-a}}_{\text{average rate of change in } [a,t]}$$

- The instantaneous rate of change at $t=a$.



$f'(a)$ is the slope of the tangent line at $t = a$

Ex 5 Compute $f'(1)$:

① $f(t) = 100 - 5t^2$

$$f'(1) = \lim_{t \to 1} \frac{f(t) - f(1)}{t - 1}$$

$$= \lim_{t \to 1} \frac{(100 - 5t^2) - (100 - 5 \cdot 1^2)}{t - 1}$$

$$= \lim_{t \to 1} \frac{-5t^2 + 5}{t - 1}$$

$$= \lim_{t \to 1} \frac{-5(t^2-1)}{t-1}$$

$$= \lim_{t \to 1} \frac{-5(t+1)(t-1)}{t-1}$$

$$= \lim_{t \to 1} \left(-5(t+1)\right) = -10$$

② $f(t) = t^3$

$$f'(1) = \lim_{t \to 1} \frac{f(t) - f(1)}{t-1}$$

$$= \lim_{t \to 1} \frac{t^3 - 1^3}{t-1}$$

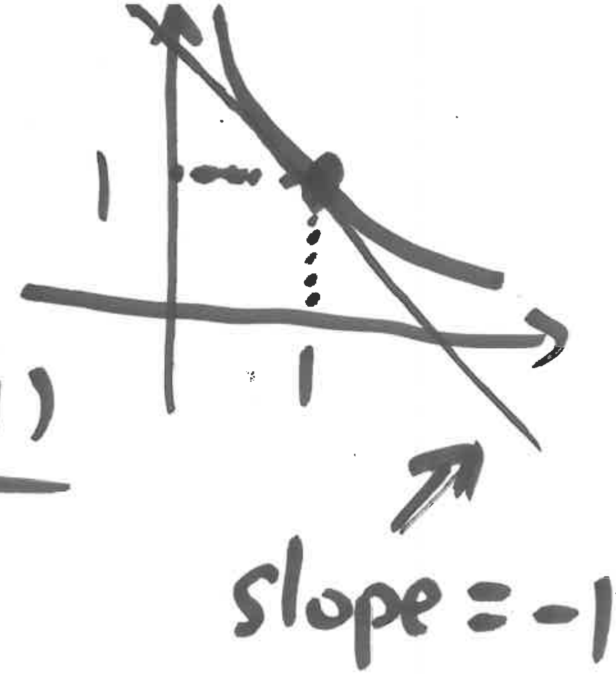
$$= \lim_{t \to 1} \frac{(t-1)(t^2+t+1)}{t-1}$$

$$= \lim_{t \to 1} (t^2+t+1) = 3$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

③ $f(t) = \frac{1}{t}$



$$f'(1) = \lim_{t \to 1} \frac{f(t) - f(1)}{t - 1}$$

$$= \lim_{t \to 1} \frac{\frac{1}{t} - 1}{t - 1}$$

$$= \lim_{t \to 1} \frac{1 - t}{(t-1) \cdot t} = \lim_{t \to 1} \frac{-1}{t}$$

$$= -1$$

Estimate derivative by average rate of change.

- By definition,

$$f'(a) \approx \frac{f(a+\delta) - f(a)}{\delta}$$

interval $[a, a+\delta]$

δ is small

ex Estimate $f'(1)$ for $f(t) = t^3$

choose interval $[1, 1+\delta]$

δ	Estimate of $f'(1)$
1	$\dfrac{2^3 - 1^3}{2-1} = 7$
0.1	$\dfrac{1.1^3 - 1^3}{1.1-1} = 3.31$
0.01	$\dfrac{1.01^3 - 1^3}{1.01-1} = 3.0301$

- A better way:

$$f'(a) \approx \frac{f(a+\delta) - f(a-\delta)}{2\delta}$$

interval $[a-\delta, a+\delta]$

<u>ex</u> $f'(1)$ for $f(t) = t^3$

(see Ex 3)

ex Estimate increasing speed of the pop. of USA at 1950, 1960, 1970, 1980

$$P'(1950) \approx \frac{P(1960) - P(1940)}{1960 - 1940}$$

$$= \frac{180.7 - 132.1}{20} = 2.43.$$

$$P'(1960) \approx \frac{P(1980) - P(1940)}{1980 - 1940}$$

$$= \frac{226.5 - 132.1}{40} = 2.36$$

$$P'(1970) = \frac{P(1980) - P(1960)}{20} = 2.29$$

$$P'(1980) = \frac{P(2000) - P(1960)}{40} = 2.54$$