Final review 3

ex 1 Conv/div of series:

① $\sum_{n=1}^{\infty} n e^{-n}$

$f(x) = x e^{-x}$

Check dec. : $f'(x) = e^{-x} - x e^{-x}$

$= (1-x) e^{-x} \leq 0$

if $x \geq 1$

$$\int_1^\infty x e^{-x}\,dx$$

$$\begin{pmatrix} u = x & v = -e^{-x} \\ du = dx & dv = e^{-x}\,dx \end{pmatrix}$$

$$= -x e^{-x}\Big|_1^\infty + \int_1^\infty e^{-x}\,dx$$

$$= e^{-1} - e^{-x}\Big|_1^\infty = 2e^{-1}$$

$\Rightarrow \int_1^\infty x e^{-x}\,dx$ Conv. $\Rightarrow \sum n e^{-n}$ conv.

② $$\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$$

$$a_n = \frac{n^n}{(2n)!} \qquad a_{n+1} = \frac{(n+1)^{n+1}}{(2(n+1))!}$$

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^{n+1} \cdot (2n)!}{(2n+2)! \cdot n^n}$$

$$= \lim_{n\to\infty} (n+1)\cdot \underbrace{\left(\frac{n+1}{n}\right)^n}_{\to e} \cdot \frac{1}{(2n+2)(2n+1)}$$

$$= 0 < 1$$

$\Rightarrow$ conv.

③ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1} = \sum_{n=1}^{\infty} (-1)^n a_n$

$$a_n = \frac{n^2}{n^2+1} \not\to 0$$

$\Rightarrow$ div.

<u>ex 2</u> Use $(\tan^{-1}(x))' = \frac{1}{1+x^2}$ to get Taylor series of $f(x) = \tan^{-1}(x)$. What's its interval of conv.?

Approx. $\tan^{-1}(0.1)$ by the first nonzero term, estimate error.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}$$

$$\Big\downarrow \text{ integrate}$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

To get R, $\sum_{n=0}^{\infty} \frac{1}{2n+1} |x|^{2n+1}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{\underbrace{\sqrt[n]{2n+1}}_{\to 1}} \cdot |x|^{\frac{2n+1}{n}}$$

$= |x|^2 < 1 \Rightarrow -1 < x < 1$

$R = 1$

at $x=1$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \cdot a_n$

$a_n = \dfrac{1}{2n+1}$ conv. (alt.)

at $x=-1$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (-1)^{2n+1}$

$\displaystyle = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ conv.

$\boxed{[-1, 1]}$

$$\tan^{-1}(0.1) \approx 0.1$$

$$r_1(0.1) = \frac{f^{(2)}(t_{0.1})}{2!} 0.1^2$$

$\uparrow$

since we stop at x^1 term

$$f(x) = \tan^{-1}(x) \qquad f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = -\frac{2x}{(1+x^2)^2}$$

$$\left| -\frac{2t_{0.1}}{(1+t_{0.1}^2)^2} \right| \le 0.2$$

$$0 < t_{0.1} < 0.1$$

$$|r_1(0.1)| \le \frac{0.2}{2} \cdot 0.1^2 = 0.001$$

ex 3 conv/div of $\int_2^{\boxed{\infty}} \frac{x}{x^3-1}\,dx$

want $\frac{x}{x^3-1} \leq \frac{2}{x^2}$ for $x \geq 2$

$$\Updownarrow$$

$$x^3 \leq 2x^3 - 2$$

$$\Updownarrow$$

$$2 \leq x^3 \quad \checkmark$$

since $\int_2^\infty \frac{2}{x^2}\,dx$ conv. ,

$\int_2^\infty \frac{x}{x^3-1}\,dx$ also conv.

$\underline{\text{ex } 4}$ $\quad r = \underbrace{1+\cos\theta}$ $\leftarrow$ always ≥ 0

Compute length, area

$$L = \int_0^{2\pi} \sqrt{r^2 + (r')^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= \cdots\cdots = 8$$

$$A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (1+\cos\theta)^2 \, d\theta .$$

$$= \frac{1}{2} \int_0^{2\pi} (1 + 2\cos\theta + \underbrace{\cos^2\theta}_{=\frac{1}{2}(1+\cos 2\theta)}) \, d\theta$$

$$= \frac{1}{2}\left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta \right)\Big|_0^{2\pi}$$

$$= \frac{3}{2}\pi .$$