Final review 2

ex 1 ① $\int x \cdot 2^x \, dx$

$= \int x \, e^{\ln(2^x)} \, dx$

$= \int x \, e^{x \cdot \ln 2} \, dx$

$$\left(\begin{array}{ll} u = x & v = \dfrac{e^{x \cdot \ln 2}}{\ln 2} \\ & \\ du = dx & dv = e^{x \cdot \ln 2} \, dx \end{array} \right)$$

$$= x \cdot \frac{e^{x \cdot \ln 2}}{\ln 2} - \int \frac{e^{x \cdot \ln 2}}{\ln 2}\, dx$$

$$= x \frac{e^{x \cdot \ln 2}}{\ln 2} - \frac{e^{x \cdot \ln 2}}{(\ln 2)^2} + C$$

② $\displaystyle\int \frac{(\log_3 x)^2}{x}\,dx$

$$= \int \left(\frac{\ln x}{\ln 3}\right)^2 \cdot \frac{1}{x}\,dx$$

$$= \frac{1}{(\ln 3)^2}\int u^2\,du$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \frac{1}{(\ln 3)^2}\cdot\frac{1}{3}u^3 + C$$

$$= \frac{1}{3(\ln 3)^2}(\ln x)^3 + C$$

$$\begin{pmatrix} u = 1 + x^2 \\ du = 2x\,dx \end{pmatrix}$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln|u| \Big|_1^2$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2$$

③ $\int_0^1 \tan^{-1}(x)\,dx$

$$\left(\begin{array}{ll} u = \tan^{-1}(x) & v = x \\ du = \dfrac{1}{1+x^2}dx & dv = dx \end{array} \right)$$

$$= x\tan^{-1}(x)\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \frac{\pi}{4} - \frac{1}{2}\int_1^2 \frac{1}{u}\,du$$

ex 2 ① $\lim_{x \to 0} \frac{x^2}{1-\cos x}$

$= \lim_{x \to 0} \frac{2x}{\sin x}$

$\frac{0}{0}$

$= \lim_{x \to 0} \frac{2}{\cos x} = 2$

(2) $\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x$

$= \lim_{x \to \infty} e^{x \ln\left(1 - \frac{1}{x}\right)}$

$= e^{\lim_{x \to \infty} x \ln\left(1 - \frac{1}{x}\right)}$

$= e^{-1}$

$$\lim_{x\to\infty} x \ln\left(1-\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\overset{\frac{0}{0}}{=} \lim_{x\to\infty} \frac{\frac{1}{1-1/x}\cdot\frac{1}{x^2}}{-\frac{1}{x^2}} = -1$$

ex 3 ① $\int \frac{1}{\sqrt{x^2-2x+5}} \, dx$

$= \int \frac{1}{\sqrt{(x-1)^2+4}} \, dx$

$= \frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{x-1}{2}\right)^2+1}} \, dx$

$\frac{x-1}{2} = \tan u$ $\qquad$ $\frac{1}{2} dx = \sec^2 u \, du$

$$= \int \frac{1}{\sec u} \cdot \sec^2 u \, du$$

$$= \int \sec u \, du$$

$$= \ln \left| \sec u + \tan u \right| + C$$

$$= \ln \left| \sqrt{\left(\frac{x-1}{2}\right)^2 + 1} + \frac{x-1}{2} \right| + C$$

② $\int \sin^2 x \cos^3 x \, dx$

$= \int \sin^2 x \cos^2 x \cdot \cos x \, dx$

$\left(\begin{array}{c} u = \sin x \\ du = \cos x \, dx \end{array} \right)$

$= \int u^2 \cdot (1 - u^2) \, du$

$= \frac{1}{3} u^3 - \frac{1}{5} u^5 + C$

$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$

③ $\int \sin^2 x \cos^2 x \, dx$

$= \int \frac{1}{2}(1-\cos 2x) \cdot \frac{1}{2}(1+\cos 2x) \, dx$

$= \frac{1}{4} \int (1-\cos^2 2x) \, dx$

$= \frac{1}{4} \int \sin^2 2x \, dx$

$= \frac{1}{4} \int \frac{1}{2}(1-\cos 4x) \, dx$

$$= \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C$$

ex 4 $\displaystyle\int \frac{x-2}{(x+1)(x^2+1)}\,dx$

$$= \int \left(\frac{A}{x+1} + \frac{Bx+C}{x^2+1} \right) dx$$

$$x-2 = A(x^2+1) + (Bx+C)(x+1)$$

$x=-1$: $-1-2 = A(1+1)$

$$A = -\frac{3}{2}$$

x^2 coeff. : $0 = A + B$

$B = -A = \frac{3}{2}$

const. coeff. : $-2 = A + C$

$C = -2 - A = -2 + \frac{3}{2} = -\frac{1}{2}$

$$\square = \int \left(\frac{-\frac{3}{2}}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1} \right) dx$$

$$= -\frac{3}{2} \ln|x+1| + \frac{3}{2} \cdot \frac{1}{2} \ln(x^2+1) - \frac{1}{2} \tan^{-1} x + C$$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + C$$

$$\int e^{-x} \cos x \, dx$$

$$= -e^{-x} \cos x \qquad \left(\begin{array}{ll} u = \cos x & v = -e^{-x} \\ du = -\sin x \, dx & dv = e^{-x} dx \end{array} \right.$$

$$- \int e^{-x} \sin x \, dx$$

$$\left(\begin{array}{ll} u = \sin x & v = -e^{-x} \\ du = \cos x \, dx & dv = e^{-x} dx \end{array} \right)$$

$$= -e^{-x} \cos x + e^{-x} \sin x$$

$$- \int e^{-x} \cos x \, dx$$

$$2\int e^{-x}\cos x\,dx = -e^{-x}\cos x + e^{-x}\sin x$$

$$\int e^{-x}\cos x\,dx = \frac{1}{2}\left(-e^{-x}\cos x + e^{-x}\sin x\right) + C$$