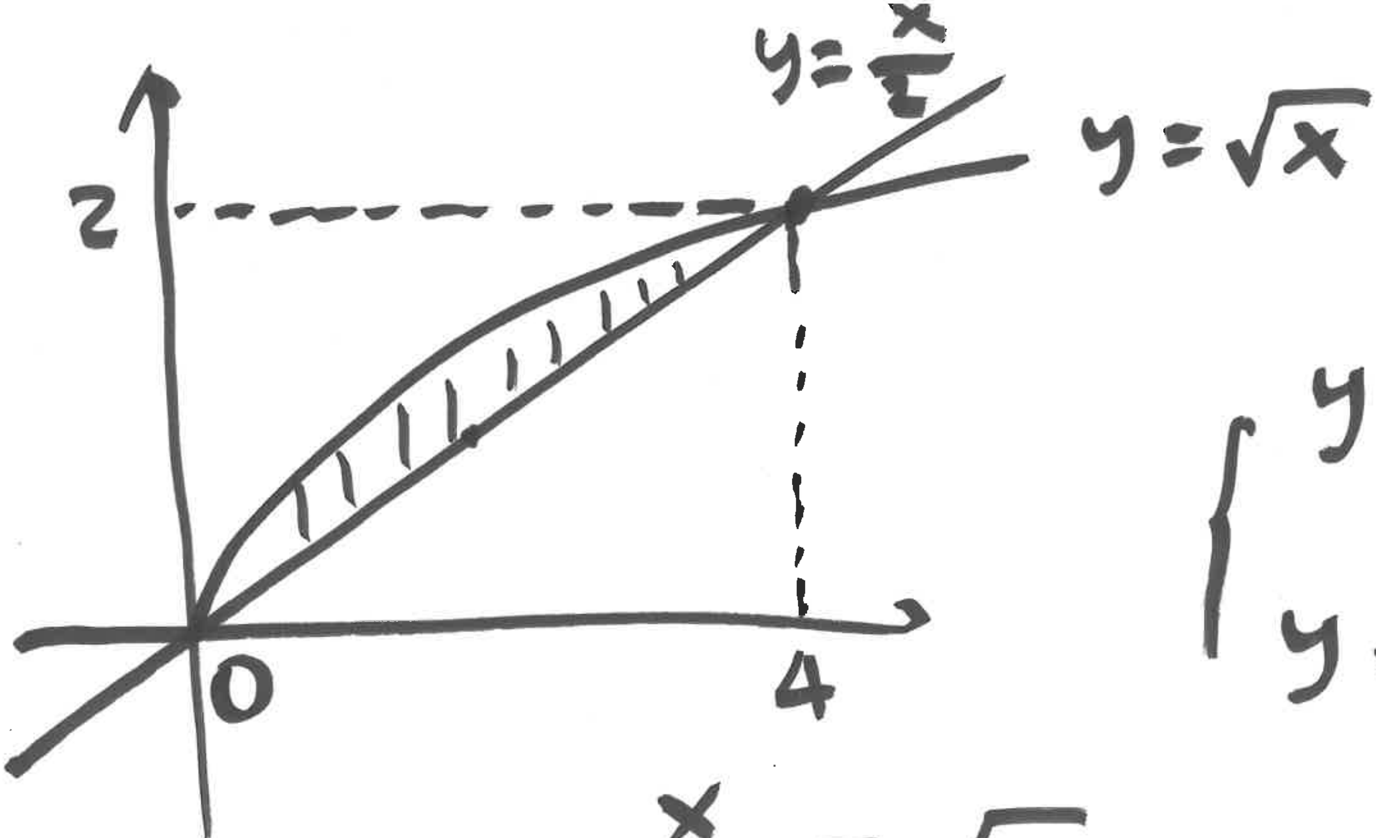
Final review 1

ex 1 Let R be the region bounded by $y = \frac{x}{2}$ and $y = \sqrt{x}$

① Volume by rotating R around x-axis

② .... y-axis

③ Center of mass of R



$$\begin{cases} y = \dfrac{x}{2} \\ y = \sqrt{x} \end{cases}$$

$$\frac{x}{2} = \sqrt{x}$$

$$\frac{x^2}{4} = x \qquad x^2 - 4x = 0$$

$$x(x-4) = 0$$

$$x = 0 \quad \text{or} \quad x = 4$$

$$y = 0 \qquad\qquad y = 2$$

① $V_x = \int_0^4 \pi\left((\sqrt{x})^2 - \left(\frac{x}{2}\right)^2\right)dx$

② $V_y = \int_0^2 \pi\left((2y)^2 - (y^2)^2\right)dy$

$y = \frac{x}{2} \rightsquigarrow x = 2y$

$y = \sqrt{x} \rightsquigarrow x = y^2$

③ $m = \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx$

$= \left(\frac{2}{3} x^{3/2} - \frac{1}{2} \cdot \frac{1}{2} x^2\right) \Big|_0^4$

$= \frac{16}{3} - 4 = \frac{4}{3}$

$M_y = \int_0^4 x \left(\sqrt{x} - \frac{x}{2}\right) dx$

$= \left(\frac{2}{5} x^{5/2} - \frac{1}{2} \cdot \frac{1}{3} x^3\right) \Big|_0^4 = \frac{32}{15}$

$$M_x = \int_0^4 \frac{1}{2}\left((\sqrt{x})^2 - \left(\frac{x}{2}\right)^2 \right) dx$$

$$= \frac{1}{2}\left(\frac{1}{2}x^2 - \frac{1}{4}\,\frac{1}{3}x^3 \right)\Big|_0^4$$

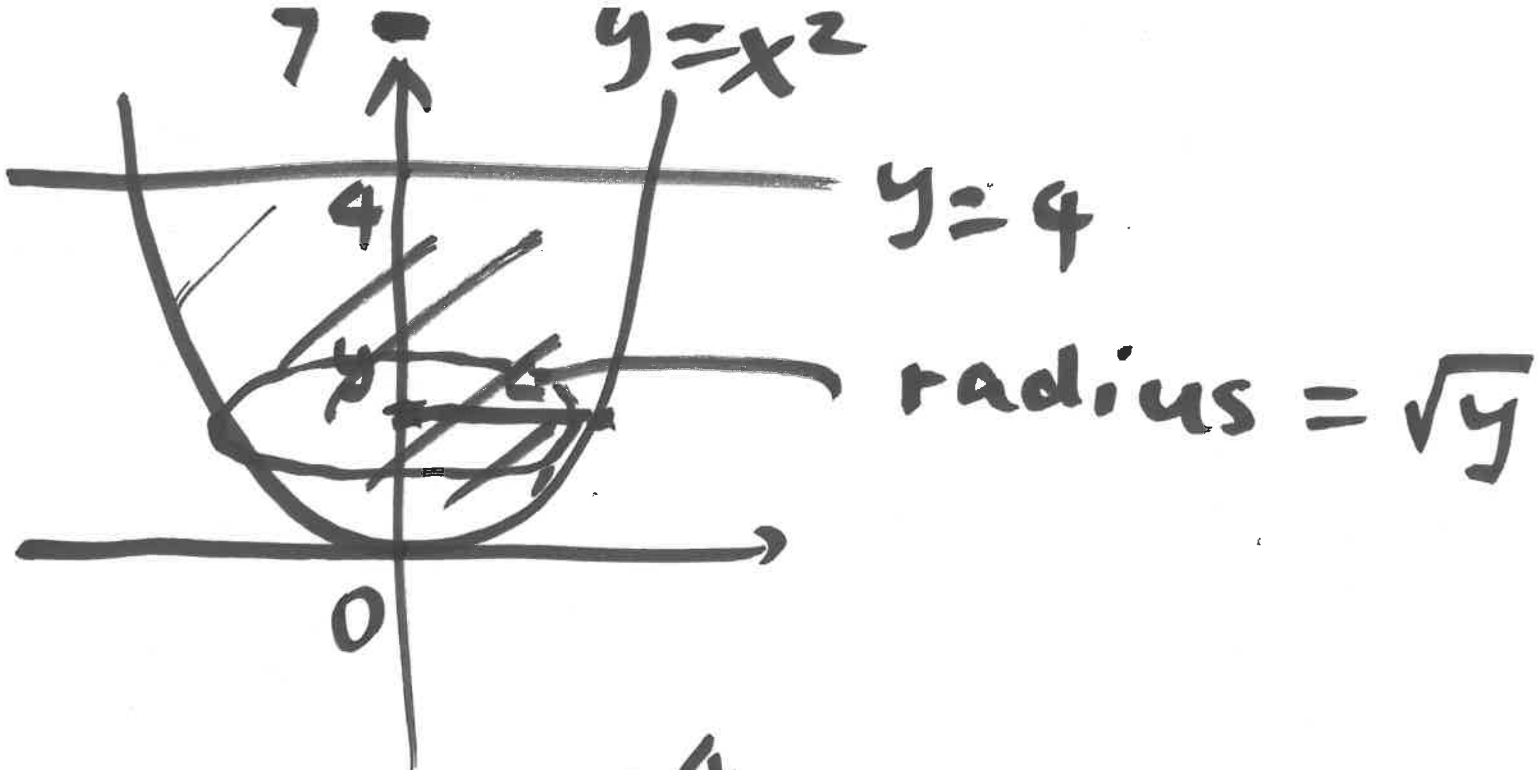
$$= 4 - \frac{64}{24} = \frac{4}{3}$$

Center of mass $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{M_y}{m} = \frac{8}{5} \qquad \bar{y} = \frac{M_x}{m} = 1$$

ex 2 A paraboloid shape water tank, formed by rotating the region bounded by $y=x^2$, $y=4$ around y-axis. It is full of water. What is the work by pumping out all water to 3 feet higher than the top?

(density of water is 62.5 lb/ft^3).



$$\text{Work} = \int_0^4 62.5 \cdot (7-y)\,\pi \cdot (\sqrt{y})^2\,dy$$

$$= 62.5\pi \int_0^4 (7y - y^2)\,dy$$

$$= 62.5\pi \left(7 \cdot \frac{1}{2} y^2 - \frac{1}{3} y^3 \right) \Big|_0^4$$

$$= 62.5\pi\left(7 \cdot \frac{1}{2} \cdot 4^2 - \frac{1}{3} 4^3\right)$$

ex 3 Find the length of the parametric curve

$$\begin{cases} x = \frac{t^2}{2} - t \\ y = \frac{4}{3} t^{3/2} \end{cases} \quad 0 \le t \le 1$$

$$L = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_0^1 \sqrt{(t-1)^2 + \left(\frac{4}{3} \cdot \frac{3}{2} t^{1/2}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{t^2 - 2t + 1 + 4t} \; dt$$

$$= \int_0^1 \sqrt{t^2 + 2t + 1} \; dt$$

$$= \int_0^1 (t+1)\, dt$$

$$= \left(\frac{t^2}{2} + t\right)\Big|_0^1 = \frac{3}{2}$$

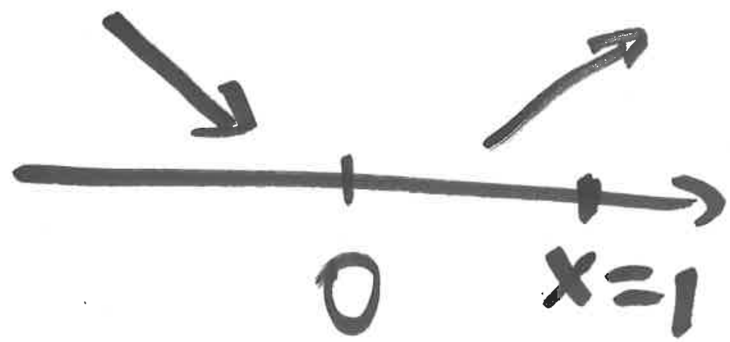
ex 4 $f(x) = e^x + e^{-x}$. Find the largest interval containing $x = 1$ such that f has an inverse. Find this inverse. What is the domain and range of f^{-1}?

$$f'(x) = e^{x} - e^{-x} = 0$$

$$e^{x} = e^{-x}$$

$$x = -x$$

$$x = 0$$



$$\boxed{[0, \infty)}$$

$$y = e^x + e^{-x}$$

$$u = e^x$$

$$y = u + \frac{1}{u}$$

$$y \cdot u = u^2 + 1$$

$$u^2 - y \cdot u + 1 = 0$$

$$u = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

$$x = \ln \frac{y \pm \sqrt{y^2 - 4}}{2}$$

take $y = 3$ $\quad x = \ln \frac{3 \pm \sqrt{5}}{2}$

$$\ln \frac{3+\sqrt{5}}{2} > 0 \quad \checkmark$$

$$\ln \frac{3-\sqrt{5}}{2} < 0$$

$$f^{-1}(x) = \ln \frac{x + \sqrt{x^2 - 4}}{2}$$

Domain of f^{-1} = range of f

$= [2, \infty)$.

range of f^{-1} = domain of f

$= [0, \infty)$

$f(0) = e^0 + e^{-0} = 2$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} (e^x + e^{-x}) = \infty$$