10.2 Length, area in polar coord.

Curve length

Curve $r = f(\theta)$, $\alpha \le \theta \le \beta$

$$\begin{cases} x = f(\theta)\cdot\cos\theta \\ y = f(\theta)\cdot\sin\theta \end{cases}$$

parametric eq. w/ parameter θ.

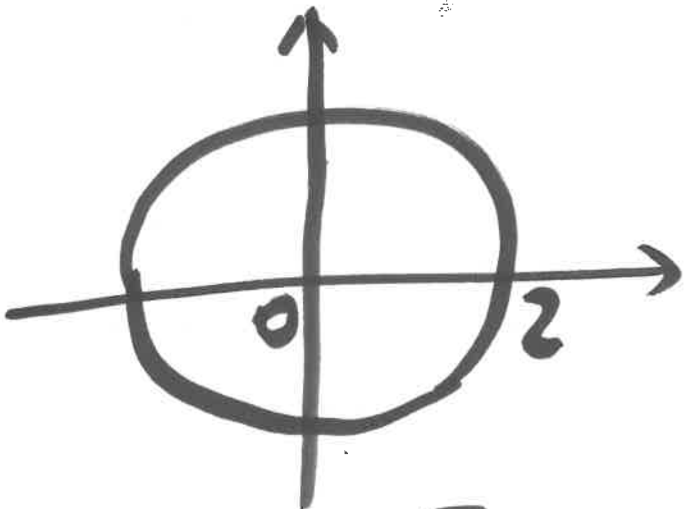
$$L = \int_\alpha^\beta \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta + f(\theta)(-\sin\theta))^2 + (f'(\theta)\sin\theta + f(\theta)\cdot\cos\theta)^2}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta$$

$$\boxed{L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

ex 1 Length of circle w/ radius $= 2$.



$r = 2$

$r' = 0$

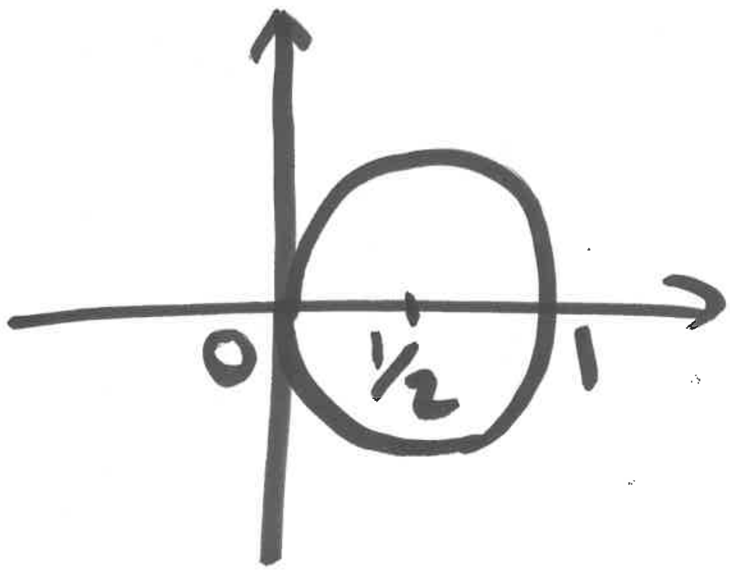
$$L = \int_0^{2\pi} \sqrt{2^2 + 0^2}\, d\theta$$

$$= \int_0^{2\pi} 2\, d\theta = 2 \cdot (2\pi - 0)$$

$$= 4\pi$$

ex 2 Length of the curve

$$r = \cos\theta \ , \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$



$$r' = -\sin\theta .$$

$$L = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(\cos\theta)^2 + (-\sin\theta)^2} \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \, d\theta = 1 \cdot \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \pi$$

ex 3 Length of the curve

$$r = \theta^2, \quad 0 \le \theta \le \pi$$

$$r' = 2\theta$$

$$L = \int_0^{\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{\theta^2 + 4}\ \ \theta\, d\theta$$

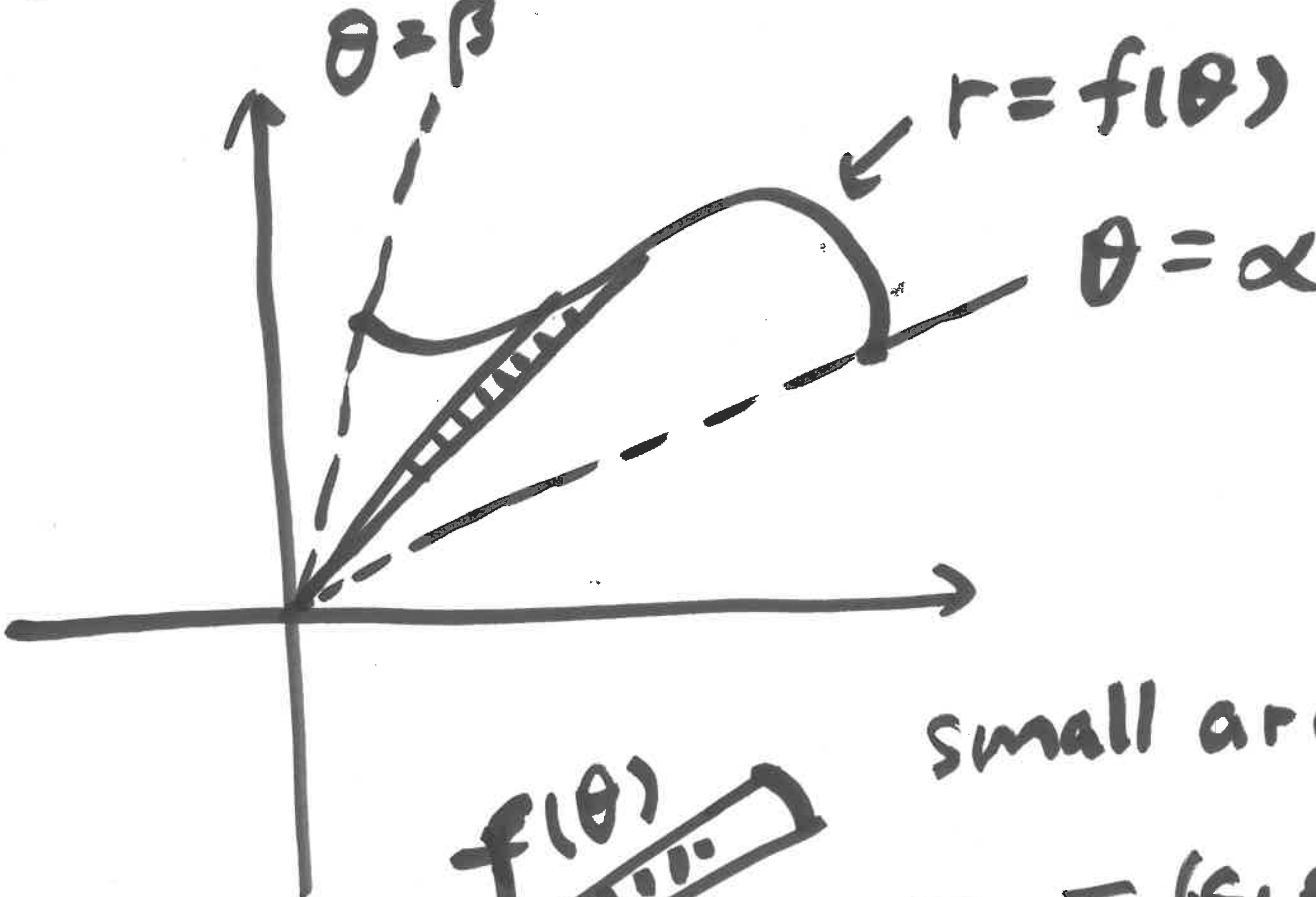
$$\begin{pmatrix} u = \theta^2 + 4 \\ du = 2\theta \, d\theta \end{pmatrix}$$

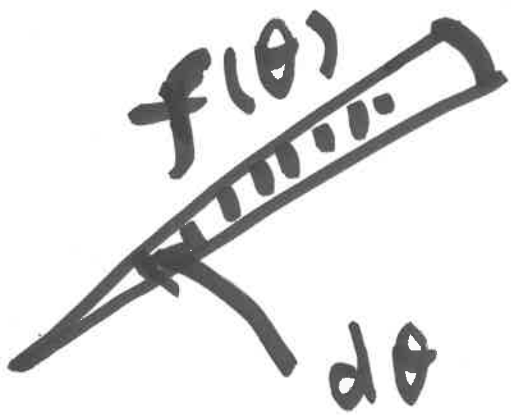
$$= \frac{1}{2} \int_4^{\pi^2+4} \sqrt{u} \, du$$

$$= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_4^{\pi^2+4}$$

$$= \frac{1}{3} \left((\pi^2+4)^{3/2} - 4^{3/2} \right)$$

Area bounded by $r = f(\theta)$, $\theta = \alpha$, $\theta = \beta$





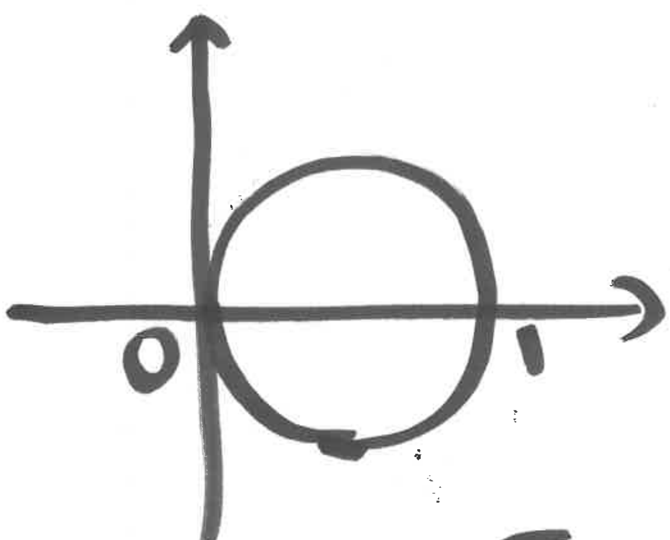
small area

$$= \pi (f(\theta))^2 \cdot \frac{d\theta}{2\pi}$$

$$= \frac{1}{2}(f(\theta))^2 \, d\theta$$

$$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2}(f(\theta))^2 \, d\theta}$$

ex 4 Area bounded by $r = \cos\theta$



$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

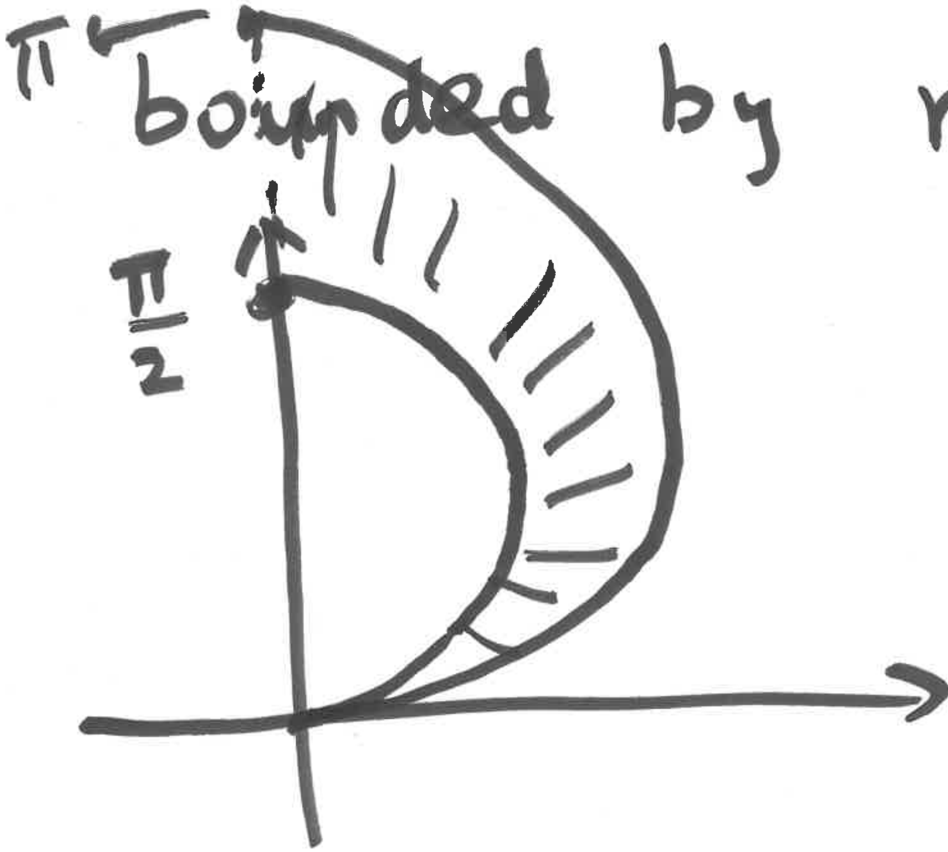
$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} \cos^2\theta \, d\theta$$

$$= \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta) \, d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{4}$$

ex 5 Area in the first quadrant bounded by $r=\theta$ and $r=2\theta$.



$$\text{Area} = \int_0^{\frac{\pi}{2}} \frac{1}{2}\left((2\theta)^2 - \theta^2\right) d\theta$$

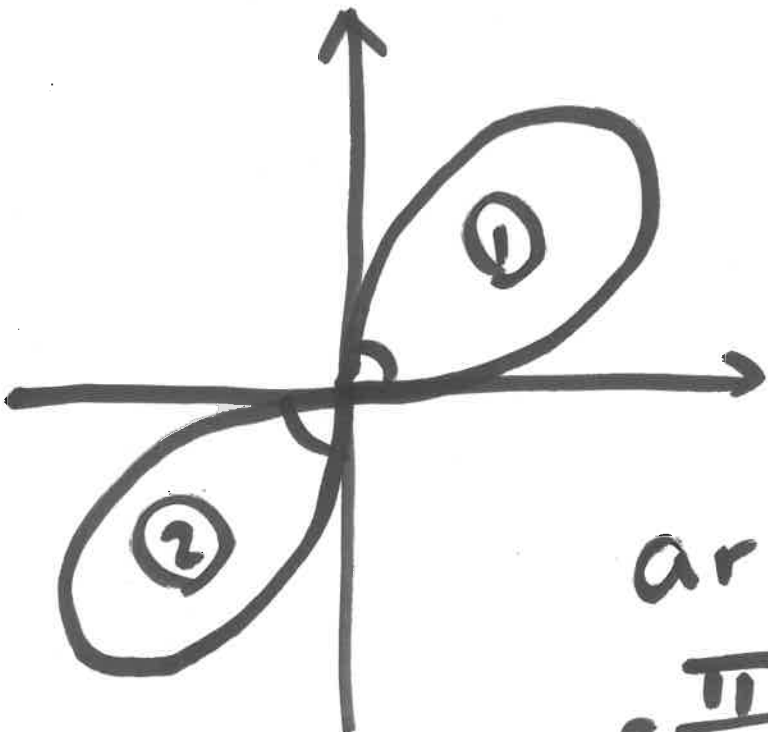
$0 \le \theta \le \frac{\pi}{2}$

$$= \frac{3}{2}\int_0^{\frac{\pi}{2}} \theta^2 \, d\theta$$

$$= \frac{3}{2} \cdot \frac{1}{3} \theta^3 \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \cdot \left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{16}$$

ex 6 Area bounded by $r^2 = 5\sin 2\theta$.



area of 1

= area of 2

area of 1

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin 2\theta \, d\theta$$

$$= -\frac{1}{4}\cos 2\theta \Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{4}\left((-1) - 1\right) = \frac{1}{2}$$

$$\text{Area} = 2 \cdot \frac{1}{2} = 1$$