SOLUTION

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name and your TA's name on every answer sheet.
(3) You can use any method to solve the problems (unless stated otherwise), but you have to justify your answers.
(4) You do not need to simplify your answer (you can leave $62.5 \cdot (10^6 - 7^6)\pi^2$ as final answer, but you cannot leave $\int_0^1 x \, \mathrm{d}x$ as final answer).
(5) You are not allowed to use calculators in this exam.

Problem 1. (25 points) Compute the following limits or series:

(1) (5 points)

$$\lim_{n\to\infty} (\sqrt{n+2} - \sqrt{n}) = \lim_{n\to\infty} \frac{(n+2)-n}{\sqrt{n+2}+\sqrt{n}} = 0$$

(2) (10 points)

$$\lim_{n\to\infty} (1+\frac{2}{n^2})^{n^2} = \lim_{x\to\infty} (1+\frac{2}{x^2})^{x^2} = \lim_{x\to\infty} e^{x^2 \ln(1+\frac{2}{x^2})} = e^{\lim_{x\to\infty} x^2 \ln(1+\frac{2}{x^2})}$$

and

$$\lim_{x\to\infty} x^2 \ln(1+\frac{2}{x^2}) = \lim_{x\to\infty} \frac{\ln(1+\frac{2}{x^2})}{1/x^2} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x^2}}(-\frac{4}{x^3})}{-2/x^3} = 2$$

by l'Hopital. So the answer is e^2.

(3) (10 points)

$$\sum_{n=0}^{\infty} 2^{-2n} = \sum_{n=0}^{\infty} 4^{-n} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

Problem 2. (30 points) Determine the convergence/divergence of the following series: (Justify your answer!)

(1) (10 points) $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$
Method 1 (limit comparison): $a_n = \frac{n}{n^3-1}$, $b_n = \frac{1}{n^2}$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^3-1} = 1 > 0$$

and $\sum b_n$ converges. Thus $\sum a_n$ converges.
Method 2 (comparison):

$$0 \le \frac{n}{n^3-1} \le \frac{n}{n^3 - n^3/2} = \frac{2}{n^2}$$

where we used $1 \leq n^3/2$ for $n \geq 2$. Since $\sum \frac{2}{n^2}$ converges, $\sum a_n$ also converges.

(2) (10 points) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

(Integral test) $f(x) = \frac{1}{x(\ln x)^2}$ is positive and decreasing, and $f(n) = \frac{1}{n(\ln n)^2}$.

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,\mathrm{d}x = \int_{\ln 2}^{\infty} \frac{1}{u^2}\,\mathrm{d}u = -\frac{1}{u}\Big|_{\ln 2}^{\infty} = \frac{1}{\ln 2} < \infty$$

This improper integral converges, and thus the series also converges.

(3) (10 points) $\sum_{n=0}^{\infty} \frac{3^n}{\sqrt{n!}}$

(Ratio test) $a_n = \frac{3^n}{\sqrt{n!}}$, $a_{n+1} = \frac{3^{n+1}}{\sqrt{(n+1)!}}$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}\sqrt{n!}}{\sqrt{(n+1)!}3^n} = \lim_{n\to\infty} \frac{3}{\sqrt{n+1}} = 0 < 1$$

Thus the series converges.

Problem 3. (25 points) Let $f(x) = \frac{1}{2-x}$.

(1) (10 points) Compute the Taylor series of $f(x)$.

$$f(x) = \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}(\frac{x}{2})^n = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}x^n$$

(2) (15 points) Use the first three terms ($n = 0, 1, 2$) of the Taylor series to approximate $f(0.1)$, and estimate its error.

The Taylor series is

$$f(x) = \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \cdots$$

Using the first three terms,

$$f(0.1) \approx \frac{1}{2} + \frac{1}{4}0.1 + \frac{1}{8}0.1^2$$

The Lagrange remainder term

$$r_2(0.1) = \frac{f^{(3)}(t_{0.1})}{3!}0.1^3, \quad 0 < t_{0.1} < 0.1$$

$f^{(3)}(x) = \frac{6}{(2-x)^4}$, thus

$$|f^{(3)}(t_{0.1})| \leq \frac{6}{(2-0.1)^4} = \frac{6}{1.9^4}$$

Thus

$$|r_2(0.1)| \leq \frac{0.1^3}{1.9^4}$$

Problem 4. (20 points)

(1) (10 points) Solve the equation $z + \frac{1}{z} = 1$ in complex numbers, and plot your solutions in the complex plane. (Write your solutions as $a + bi$, where a and b are real numbers.)

$$z^2 - z + 1 = 0, \quad z = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

(You can write your answer as either $\frac{1 \pm \sqrt{3}i}{2}$ or $\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$.) The corresponding two points in the complex plane are $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$. (They lie in the first and fourth quadrants, respectively, and are symmetric around the real axis. Picture omitted here.)

(2) (10 points) Determine the convergence/divergence of the series $\sum_{n=1}^{\infty} \frac{\cos(3\pi n)}{n}$. (Justify your answer!)

Notice that $\cos(3\pi n) = (-1)^n$ for integers n. Thus this series converges by the alternating series test ($\frac{1}{n}$ is positive, decreasing, and going to zero).