Chapter 9 review

ex 1 $$\lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{x\to\infty} x \sin\frac{1}{x}$$

$$= \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \underset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to\infty} \frac{\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = 1$$

$$\sum_{n=2}^{\infty} \frac{1}{3^n} = \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots$$

$$= \frac{1}{3^2}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\right)$$

$$= \frac{1}{3^2}\,\frac{1}{1-\frac{1}{3}} = \frac{1}{9}\cdot\frac{3}{2} = \frac{1}{6}.$$

$$1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

$$1 + a + a^2 + \cdots = \lim_{n \to \infty} \frac{1 - a^{n+1}}{1 - a}$$

$\underline{|a| < 1}$

$$= \frac{1}{1 - a}$$

ex2 Determine conv/div :

① $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \int_{\ln 2}^{\infty} \frac{1}{\sqrt{u}}\,du$$

$$\left(u = \ln x \qquad du = \frac{1}{x}\,dx \right)$$

$$= 2\sqrt{u}\,\Big|_{\ln 2}^{\infty} = \infty \quad \text{div.}$$

$\Longrightarrow$ integral test $\quad \displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ $\boxed{\text{div}}$

(2) $$\sum_{n=1}^{\infty} \frac{\sin n}{n^3+1}$$

$$\left| \frac{\sin n}{n^3+1} \right| \le \frac{1}{n^3+1} \le \frac{1}{n^3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ conv.} \Rightarrow \sum_{n=1}^{\infty} \frac{\sin n}{n^3+1} \text{ conv.}$$

Comparison.

③ $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.1}}$$

$$\sum_{n=1}^{\infty} (-1)^n \cdot a_n \quad , \quad a_n = \frac{1}{n^{0.1}}$$

$$a_n > 0 \quad , \quad a_n \text{ dec.} \quad , \quad \lim_{n\to\infty} a_n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n a_n \text{ conv.}$$

(alternating)

④ $\sum_{n=1}^{\infty} \frac{2^n}{n^6}$

$a_n = \frac{2^n}{n^6}$ $\qquad$ $a_{n+1} = \frac{2^{n+1}}{(n+1)^6}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^6} \frac{n^6}{2^n}$

$= \lim_{n\to\infty} 2 \cdot \left(\frac{n}{n+1}\right)^6 = 2 > 1$

$\Rightarrow$ div. (ratio test).

$$a_n = \frac{2^n}{n^6}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{2}{(\sqrt[n]{n})^6} = 2 > 1$$

$\Rightarrow$ div. (root test)

⑤ $$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$$

$$a_n = \frac{3\cdot 5\cdots\cdots(2n-1)}{2\cdot 4\cdots\cdots(2n-2)} \cdot \frac{1}{2n}$$

$$= \frac{3}{2}\cdot\frac{5}{4}\cdots\cdots\frac{2n-1}{2n-2}\cdot\frac{1}{2n}$$

$$> \frac{1}{2n}, \quad \sum_{n=1}^{\infty}\frac{1}{2n} \text{ div}$$

$$\xRightarrow{\text{comp.}} \sum_{n=1}^{\infty} a_n \; \boxed{\text{div}}$$

ex 3. $f(x) = e^{-3x}$

Compute Taylor series, its radius of conv, approx. $f(0.1)$ by $n=0,1,2$ terms, estimate error

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$e^{-3x} = \sum_{n=0}^{\infty} \frac{1}{n!} (-3x)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^n$$

$$\sum_{n=0}^{\infty} \frac{3^n}{n!} |x|^n$$

$$\lim_{n\to\infty} \frac{3^{n+1} \cdot |x|^{n+1}}{(n+1)!} \frac{n!}{3^n \cdot |x|^n}$$

$$= \lim_{n\to\infty} \frac{3 \cdot |x|}{(n+1)} = 0 < 1$$

conv for all x (ratio test)

$R = \infty$

$$f(0.1) \approx 1 - 3 \cdot 0.1 + \frac{9}{2} 0.1^2$$

$$r_2(0.1) = \frac{f^{(3)}(t_{0.1})}{3!} 0.1^3$$

$$f^{(3)}(x) = (-3)^3 e^{-3x} = -27e^{-3x}$$

$$|f^{(3)}(t_{0.1})| = 27e^{-3t_{0.1}} \le 27$$

$$0 < t_{0.1} < 0.1$$

$$|r_2(0.1)| \le \frac{27}{6} 0.1^3$$

ex 4 Solve $z^2 - 2z + 2 = 0$ in complex numbers.

$$z = \frac{2 \pm \sqrt{4 - 4\cdot1\cdot2}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$