Complex numbers

Euler's formula

$$e^{iz} = \cos z + i \sin z$$

Proof:
$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$$

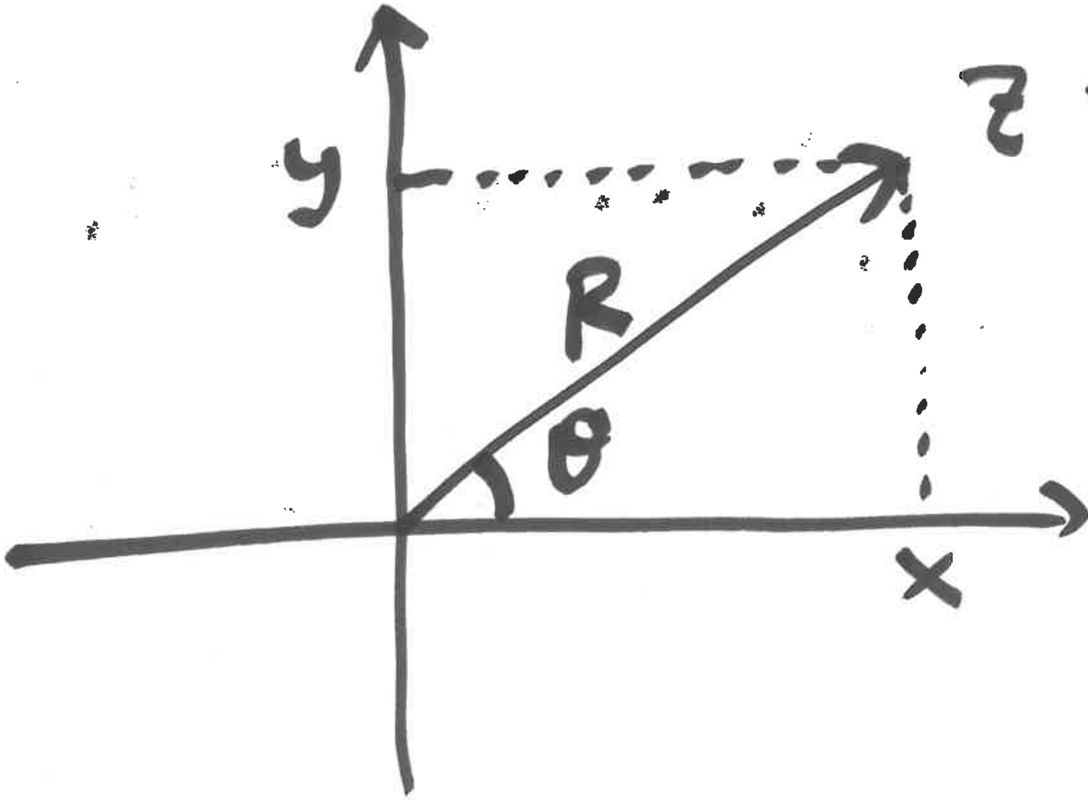
$$= 1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \cdots$$

$$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\cdots\right)$$

$$+ i\left(z - \frac{z^3}{3!} + \cdots\cdots\right)$$

$$= \cos z + i \sin z$$

Polar form



$z = x + iy$

$R = |z|$

$x = R\cos\theta$

$y = R\sin\theta$

$$z = R\cos\theta + iR\sin\theta = \underline{Re^{i\theta}}$$

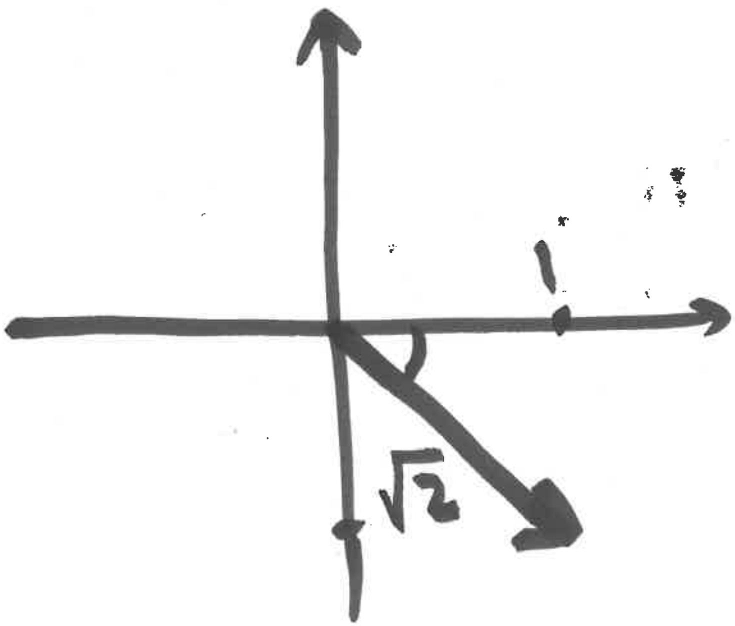
$$z_1 = R_1 e^{i\theta_1} , \quad z_2 = R_2 e^{i\theta_2}$$

$$z_1 \cdot z_2 = R_1 R_2 e^{i(\theta_1 + \theta_2)}$$

$$z_1^n = R_1^n e^{in\theta_1}$$

<u>ex 1</u> Compute $(1-i)^{10}$

$$R = |1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$$



$$\theta \in \left[-\frac{\pi}{2}, 0\right]$$

$$\cos\theta = \frac{1}{\sqrt{2}}$$

$$\theta = -\frac{\pi}{4}$$

$$1 - i = \sqrt{2}\, e^{i\cdot(-\frac{\pi}{4})}$$

$$(1-i)^{10} = (\sqrt{2})^{10}\, e^{i(-\frac{\pi}{4})\cdot 10}$$

$$= 32\, e^{-i\frac{5\pi}{2}}$$

$$= 32\left(\cos\left(-\frac{5\pi}{2}\right) + i\sin\left(-\frac{5\pi}{2}\right)\right)$$

$$= 32(0 + i(-1))$$

$$= -32i$$

<u>ex 2</u> Solve $z^n = 1$.

$z = Re^{i\theta}$ $\quad R^n e^{in\theta} = 1$

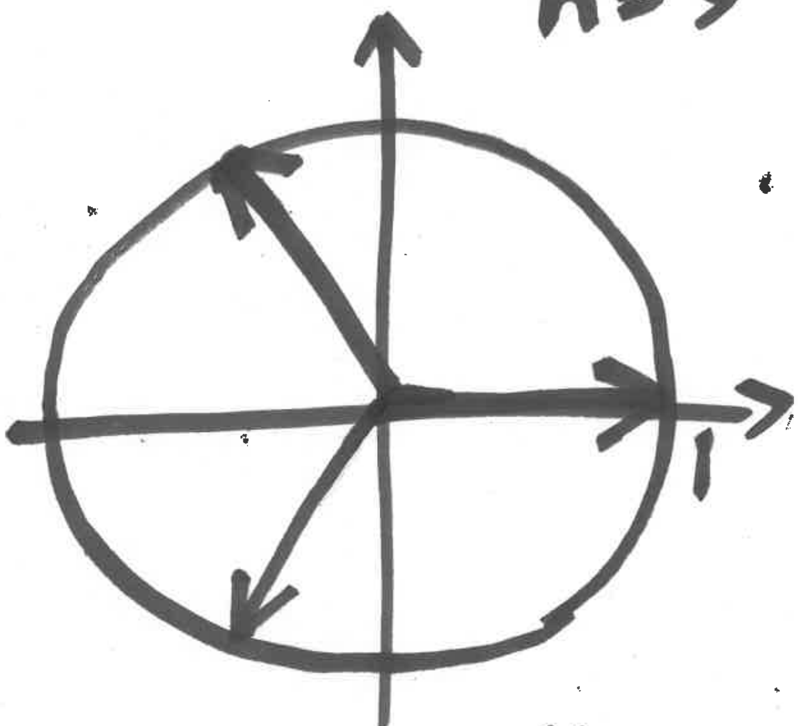
modulus: $R^n = 1 \Rightarrow R = 1$

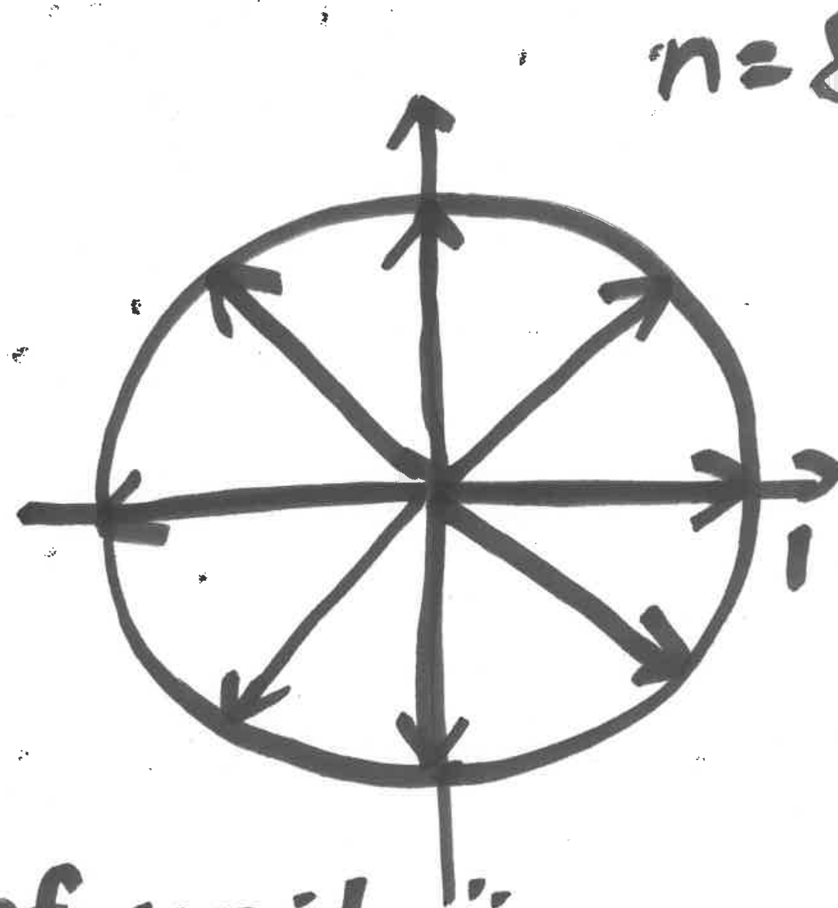
angle: $n\theta = 0 + 2k\pi$

k integer.

$$\theta = \frac{2k\pi}{n} \quad k = 0, 1, \cdots, n-1$$

$$z = e^{i \cdot \frac{2k\pi}{n}} \qquad k = 0, 1, \cdots, n-1$$





"roots of unity"

ex 3 $\displaystyle\int e^{(a+bi)x}\,dx$

$$= \frac{e^{(a+bi)x}}{a+bi} + C$$

$$e^{(a+bi)x} = e^{ax} e^{ibx}$$

$$= e^{ax}(\cos bx + i \sin bx)$$

$$\frac{e^{(a+bi)x}}{a+bi}$$

$$= \frac{e^{ax}(\cos bx + i\sin bx)}{a+bi}$$

$$= \frac{e^{ax}(\cos bx + i\sin bx)(a-bi)}{(a+bi)(a-bi)}$$

$$= \frac{e^{ax}}{a^2+b^2}\Big[(a\cos bx + b\sin bx)$$

$$+ i(a\sin bx - b\cos bx)\Big]$$

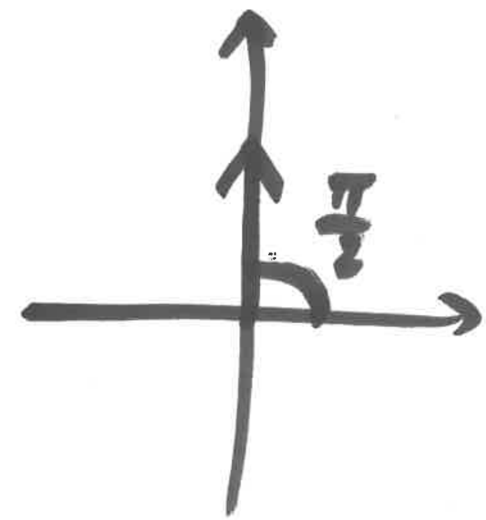
real part: $\int e^{ax}\cos bx\,dx$

$$= \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C.$$

<u>ex 4</u> Solve $z^3 = 2i$

$$z = Re^{i\theta}$$

$$R^3 e^{i \cdot 3\theta} = 2e^{i\frac{\pi}{2}}$$



modulus : $R^3 = 2$

$$R = \sqrt[3]{2}$$

angle : $3\theta = \frac{\pi}{2} + 2k\pi$

$$\theta = \frac{\pi}{6} + \frac{2k\pi}{3} \qquad k = 0, 1, 2$$

$$= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6}$$