9.9 Taylor series

The Taylor series of $f(x)$

is $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

"Maclaurin"

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

is true if $f(x)$ is defined by a power series.

Taylor's Theorem

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(t_x)}{(n+1)!}x^{n+1}$$

t_x is between 0 and x

Lagrange remainder

$r_n(x)$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\Longleftrightarrow \quad r_n(x) \to 0 \quad \text{as } n \to \infty.$$

<u>ex 1</u> $f(x) = e^x$

Taylor series is $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$$r_n(x) = \frac{f^{(n+1)}(t_x)}{(n+1)!} x^{n+1}$$

$$= \frac{e^{t_x}}{(n+1)!} x^{n+1}$$

Fix x.

① when $x > 0$, $0 < t_x < x$

$$|e^{t_x}| \leq e^x$$

$$|r_n(x)| \leq \frac{e^x}{(n+1)!} x^{n+1} \to 0$$

as $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} r_n(x) = 0$$

② $x < 0$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n .$$

...

estimate error

$$e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} \approx \sum_{n=0}^{3} \frac{1}{n!}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} = 2.666\ldots$$

$$\text{error} = r_3(1) = \frac{e^{t_1}}{4!}$$

$$0 < t_1 < 1$$

$$|r_3(1)| \le \frac{e^1}{4!} = \frac{e}{24}$$

* $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$

ex 2 "subs. trick"

$$e^{2x} = \sum_{n=0}^{\infty} \frac{1}{n!}(2x)^n$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$$

$$\sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x^3)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{6n+3}$$

$$= x^3 - \frac{1}{6} x^9 + \frac{1}{120} x^{15} - \dots - .$$

$$\frac{x^2}{1+2x} = x^2 \cdot \frac{1}{1-(-2x)}$$

$$= x^2 \cdot \sum_{n=0}^{\infty} (-2x)^n$$

$$= \sum_{n=0}^{\infty} (-2)^n x^{n+2}$$

Taylor series about $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

<u>ex 3</u> Taylor series of $f(x) = \frac{1}{x}$ about $x = 2$.

$$f(x) = \frac{1}{x} = \frac{1}{(x-2)+2}$$

$$= \frac{1}{2} \cdot \frac{1}{1 + \frac{x-2}{2}}$$

$$= \frac{1}{2} \cdot \frac{1}{1 - \left(-\frac{x-2}{2}\right)}$$

$$= \frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(-\frac{x-2}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{n+1}} (x-2)^n$$

ex 4 Taylor series of $f(x)=\ln x$ about $x=1$

$$\left(\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}\right)$$

$$f(x)=\ln x = \ln((x-1)+1)$$

$$= \ln\left(1-(-(x-1))\right)$$

$$= -\sum_{n=1}^{\infty} \frac{(-(x-1))^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$$