9.8 Power series

Power series: $\sum_{n=0}^{\infty} c_n x^n$

$= c_0 + c_1 x + c_2 x^2 + \cdots$

<u>Theorem</u> Given $\sum_{n=0}^{\infty} c_n x^n$, there exists a unique $R \in [0, \infty]$ such that

- $\sum c_n x^n$ conv. abs. for $|x| < R$
- $\sum c_n x^n$ div. for $|x| > R$

$R = 0 \Rightarrow \sum c_n x^n$ only conv. for $x = 0$

$R = \infty \Rightarrow \sum c_n x^n$ conv abs for all x.

$x = \pm R$: $\sum c_n x^n$ may conv or div

- R: radius of conv.

ex 1 $\quad \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\left(c_n = \dfrac{1}{n!} \right)$

$a_n = \dfrac{x^n}{n!} \qquad a_{n+1} = \dfrac{x^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{x^{n+1}}{(n+1)!}}{\dfrac{x^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{x^{n+1} \cdot n!}{(n+1)! \cdot x^n} = \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1$$

for all x

$$\Rightarrow \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ conv.} \quad \boxed{R = \infty}$$

ex 2 Determine interval of conv. for $\sum_{n=1}^{\infty} \frac{3^n x^n}{\sqrt{n}}$

① get R : assume $x > 0$

$$a_n = \frac{3^n x^n}{\sqrt{n}}$$

root test

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{3x}{(\sqrt[n]{n})^{1/2}} = 3x$$

$3x < 1$ when $x < \frac{1}{3}$ conv.

$3x > 1$ when $x > \frac{1}{3}$ div.

$R = \frac{1}{3}$

② $x = \frac{1}{3}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ div.

$x = -\frac{1}{3}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ conv.

(by alt. series $\sum (-1)^n a_n$

$a_n = \frac{1}{\sqrt{n}}$)

Interval of conv.:

$$\left[-\frac{1}{3}, \frac{1}{3}\right)$$

Theorem

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n x^n\right) = \sum_{n=0}^{\infty} c_n \frac{d}{dx}(x^n)$$

$$= \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad |x| < R$$

R : radius of Conv. of $\sum c_n x^n$.

has the same radius of conv.

$$\int_0^x \left(\sum_{n=0}^{\infty} c_n t^n\right) dt = \sum_{n=0}^{\infty} c_n \int_0^x t^n \, dt$$

$$= \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1} \qquad |x| < R$$

ex 3

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R = \infty$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

$$= f(x)$$

$$f(0) = 1 \qquad \text{“} \Longrightarrow \text{”} \quad f(x) = e^x$$

ex 4 $\quad f(x) = \sum_{n=0}^{\infty} x^n \qquad R = 1$

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \int_0^x t^n\,dt$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\Rightarrow -\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$|x| < 1$

$|x| < 1$

$$\sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1 - x}$$

$$\lim_{N \to \infty} \sum_{n=0}^{N} x^n = \frac{1}{1-x}$$

$$\int_0^x \frac{1}{1-t}\,dt = -\ln|1-t|\Big|_0^x$$

$$= -\ln(1-x)$$