9.7 Alternating series, absolute convergence

October 31, 2018

Alternating series: $\sum(-1)^n a_n$ or $\sum(-1)^{n+1}a_n$ with $a_n > 0$. Something like

$$a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$$

Theorem 0.1. *Let $\{a_n\}$ be positive, decreasing, and $\lim_{n\to\infty} a_n = 0$. Then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.*

The j-th truncation error $E_j := |\sum_{n=j+1}^{\infty} a_n| \leq a_{j+1}$.

Explain the 'proof' by the case $a_n = \frac{1}{n}$:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots$$

Take $j = 5$.

$$\text{first,} \quad \sum_{n=6}^{\infty} = -(\frac{1}{6} - \frac{1}{7}) - (\frac{1}{8} - \frac{1}{9}) - (\frac{1}{10} - \frac{1}{11}) - \cdots < 0$$

$$\text{second,} \quad \sum_{n=6}^{\infty} = -\frac{1}{6} + (\frac{1}{7} - \frac{1}{8}) + (\frac{1}{9} - \frac{1}{10}) + \cdots > -\frac{1}{6}$$

This shows $-\frac{1}{6} < \sum_{n=6}^{\infty} < 0$, that is, $E_5 < a_6 = \frac{1}{6}$.

Since $a_n \to 0$, E_j is small for large j, this basically (although not rigorous) implies the convergence of the series itself.

Example 1 Show $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ converges and give a bound for E_{100}.

Here $a_n = \frac{1}{\ln n} > 0$, decreasing, and going to zero. Thus the theorem implies that the alternating series converges.

$$E_{100} \leq a_{101} = \frac{1}{\ln 101}$$

Theorem 0.2. *If $\sum|a_n|$ converges, then $\sum a_n$ converges.*

Interpretation: if the total size of those numbers is finite, then the series always converges no matter how you oscillate them.

Its converse is not true: for example, $\sum\frac{(-1)^n}{n}$ converges (by alternating series) but $\sum\frac{1}{n}$ diverges.

Definition: When $\sum|a_n|$ converges, we say $\sum a_n$ converges absolutely. When $\sum|a_n|$ diverges but $\sum a_n$ converges, we say $\sum a_n$ converges conditionally.

Sometimes you are required to determine whether $\sum a_n$ converges absolutely or converges conditionally or diverges. You first do the convergence test of $\sum|a_n|$ (the generalized convergence test). In case it diverges, you further look at $\sum a_n$ itself.

Example 2 Determine abs convergence / cond convergence / divergence:

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

$$|\frac{\sin n}{n^2}| \leq \frac{1}{n^2}, \quad \sum \frac{1}{n^2} \text{ converges}$$

Thus $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely (by comparison).

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$$|\frac{(-1)^n}{\sqrt{n}}| = \frac{1}{\sqrt{n}}, \quad \sum \frac{1}{\sqrt{n}} \text{ diverges}$$

This means $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ does NOT converge absolutely. Then we notice that it is alternating, with $a_n = \frac{1}{\sqrt{n}} > 0$, decreasing, going to zero. Thus $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges. Therefore the conclusion is that $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges conditionally.

$$\sum_{n=1}^{\infty} \frac{n(-1)^n}{n+1}$$

It diverges since $a_n = \frac{n(-1)^n}{n+1}$ doesn't converge to zero.

1 9.8 Power series

Power series: a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

It is a generalization of polynomial: 'polynomial of infinite degree'.

Question, given a power series, for what values of x does it converge?

Theorem 1.1. *Given a power series $\sum_{n=0}^{\infty} c_n x^n$, there exists a unique $R \in [0, \infty]$ such that*

- $\sum_{n=0}^{\infty} c_n x^n$ *converges absolutely for every $|x| < R$.*
- $\sum_{n=0}^{\infty} c_n x^n$ *diverges for every $|x| > R$.*

R is called the radius of convergence of this power series.

When $R = 0$, the series only converges for $x = 0$.

When $R = \infty$, the series converges for all R.

At $x = \pm R$, the series may converge or diverge.

Usually you need ratio test or root test to compute R.

Example 3 Compute R for $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Use ratio test:

$$a_n = \frac{x^n}{n!}, \quad a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \lim_{n\to\infty} \frac{x}{n+1} = 0, \quad \text{for any } x$$

This means $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for any x. Thus $R = \infty$.