9.6 Positive series, ratio test, root test

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Theorem 0.1. *(Ratio test) Let $\sum a_n$ be a positive series (all $a_n > 0$) and*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$$

(possibly infinity). Then

- *If $0 \le r < 1$, then $\sum a_n$ converges.*
- *If $r > 1$ (including ∞), then $\sum a_n$ diverges.*

When $r = 1$, the theorem tells nothing!
The proof is basically a comparison test between $\sum r^n$ (don't need to go over).
Ratio test is useful when you see a^n or $n!$.

Example 1

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

$$a_n = \frac{n}{2^n}, \quad a_{n+1} = \frac{n+1}{2^{n+1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{(n+1)2^n}{n2^{n+1}} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1$$

The series converges.

Example 2

$$\sum_{n=1}^{\infty} \frac{n!}{2^n}$$

$$a_n = \frac{n!}{2^n}, \quad a_{n+1} = \frac{(n+1)!}{2^{n+1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}} = \lim_{n\to\infty} \frac{(n+1)!2^n}{n!2^{n+1}} = \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1$$

The series diverges.

Example 3

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

If use ratio test,

$$a_n = \frac{n}{n^2+1}, \quad a_{n+1} = \frac{n+1}{(n+1)^2+1}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{n+1}{(n+1)^2+1}}{\frac{n}{n^2+1}}=\lim_{n\to\infty}\frac{((n+1)!2^n)n+1)(n^2+1)}{n((n+1)^2+1)}=1$$

The ratio test tells nothing!

When n becomes large, the 1 in the denominator is negligible. So by using limit comparison with $\sum\frac{1}{n}$:

$$\lim_{n\to\infty}\frac{\frac{n}{n^2+1}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n^2}{n^2+1}=1$$

$\sum\frac{1}{n}$ diverges, limit comparison gives: $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ diverges.

Theorem 0.2. *(Root test) Let $\sum a_n$ be a positive series and*

$$\lim_{n\to\infty}\sqrt[n]{a_n}=r$$

(possibly infinity). Then

- *If $0\leq r<1$, then $\sum a_n$ converges.*
- *If $r>1$ (including ∞), then $\sum a_n$ diverges.*

When $r=1$, the theorem tells nothing!

Root test is useful when you see a^n (but not $n!$). It is not as useful as ratio test, but sometimes calculation is simpler.

Example 4

$$\sum_{n=1}^{\infty}\frac{1}{(n+3)^n}$$

$$a_n=\frac{1}{(n+3)^n}$$

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\frac{1}{(n+3)^n}}=\lim_{n\to\infty}\frac{1}{n+3}=0<1$$

The series converges.

Example 5

$$\sum_{n=1}^{\infty}\frac{n}{2^n}$$

(same as example 1)

$$a_n=\frac{n}{2^n}$$

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\frac{n}{2^n}}=\lim_{n\to\infty}\frac{\sqrt[n]{n}}{2}=\frac{1}{2}<1$$

(where $\lim_{n\to\infty}\sqrt[n]{n}=1$, always remember this!) The series converges.

Example 6

$$\sum_{n=1}^{\infty}\frac{n!}{n^n}$$

$$a_n=\frac{n!}{n^n},\quad a_{n+1}=\frac{(n+1)!}{(n+1)^{n+1}}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!n^n}{n!(n+1)^{n+1}}=\lim_{n\to\infty}(\frac{n}{n+1})^n=\lim_{n\to\infty}(1-\frac{1}{n+1})^n=e^{-1}<1$$

The series converges.