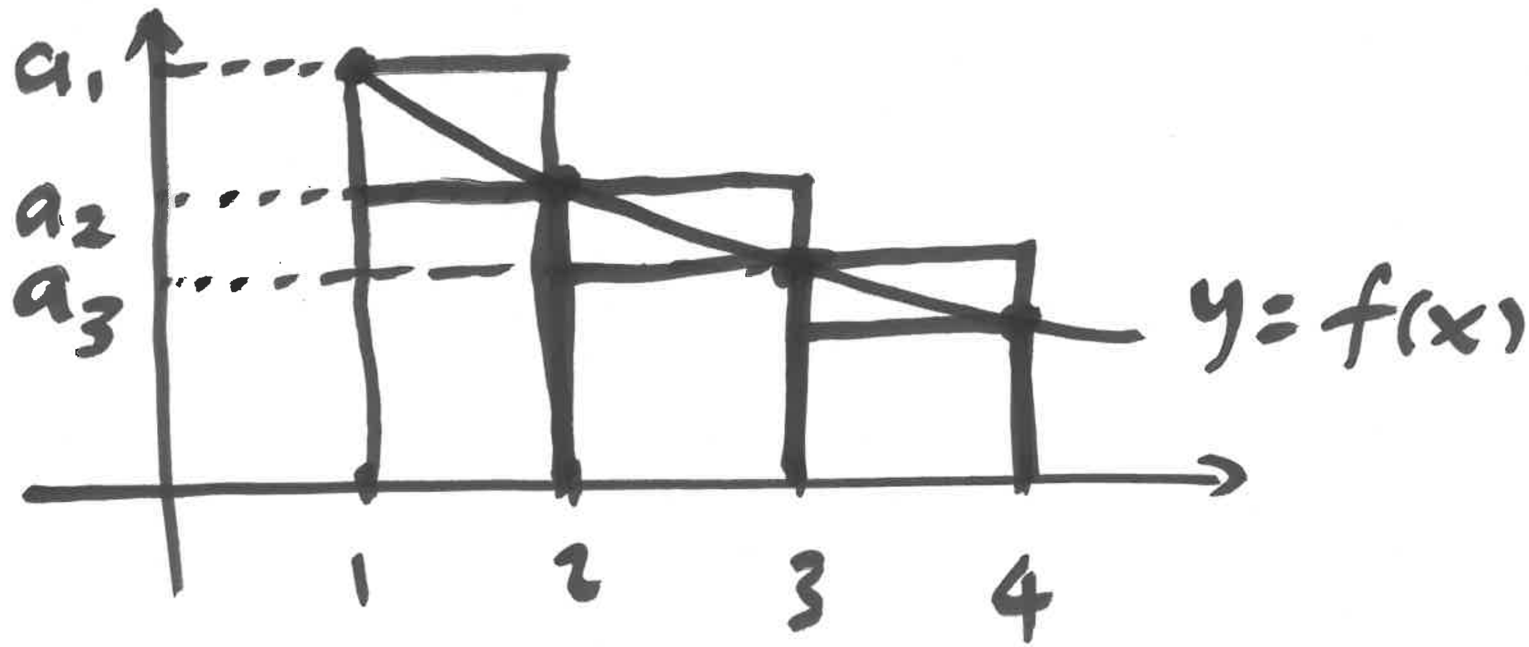
9.5 Positive series: integral test, comparison test

Positive series: $\sum a_n$, $a_n > 0$.



<u>Theorem (integral test)</u>

$\sum_{n=1}^{\infty} a_n$ positive series, a_n decreasing

If $f(x)$ decreasing on $[1, \infty)$

$f(n) = a_n \quad , n \geq 1$

then $\sum_{n=1}^{\infty} a_n$ conv $\Leftrightarrow \int_1^{\infty} f(x)\,dx$ conv.

ex 1 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\quad (p > 0)$

$$a_n = \frac{1}{n^p}$$

$$f(x) = \frac{1}{x^p} \qquad x \in [1, \infty)$$

dec. ✓

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ conv.} \iff \int_1^{\infty} \frac{1}{x^p}\,dx \text{ conv.}$$

$$\int_1^\infty \frac{1}{x^p}\,dx \qquad \text{conv.} \quad p > 1$$

$$\text{div.} \quad p \le 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p} \qquad \text{conv.} \quad p > 1$$

$$\text{div.} \quad p \le 1$$

"p-series"

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Error estimate. $\sum_{n=1}^{\infty} a_n$

$$E_j = \sum_{n=j+1}^{\infty} a_j$$

$$\int_{j+1}^{\infty} f(x)\,dx \le E_j \le \int_{j}^{\infty} f(x)\,dx$$

$f(x)$: as in integral test.

ex 2 Estimate E_{10} for $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$f(x) = \frac{1}{x^3}$ $\qquad j = 10$

$$\int_{10}^{\infty} \frac{1}{x^3}\,dx = -\frac{1}{2} \cdot \frac{1}{x^2}\Big|_{10}^{\infty} = \frac{1}{200}$$

$$\int_{11}^{\infty} \frac{1}{x^3}\,dx = -\frac{1}{2} \cdot \frac{1}{x^2}\Big|_{11}^{\infty} = \frac{1}{242}$$

$$\frac{1}{242} \le E_{10} \le \frac{1}{200}$$

Theorem (comparison)

If $0 \le a_n \le b_n$ then

- if $\sum b_n$ conv., then $\sum a_n$ conv.
- if $\sum a_n$ div., then $\sum b_n$ div.

Theorem (limit comparison)

If $a_n, b_n \ge 0$, $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$

$0 < L < \infty$. Then $\sum a_n$ conv $\Leftrightarrow$ $\sum b_n$ conv.

ex 3 $\displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{n^3+n+1}$

① Comparison.

$$0 \le \frac{2n+5}{n^3+n+1} \le \frac{2n+5n}{n^3} = \frac{7}{n^2}$$

$n \ge 1$ $\displaystyle\sum_{n=1}^{\infty} \frac{7}{n^2}$ Conv.

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{n^3+n+1}$ Conv.

② limit comparison.

$$a_n = \frac{2n+5}{n^3+n+1} \qquad b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n+5}{n^3+n+1}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{n^2(2n+5)}{n^3+n+1} = 2$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. $\Rightarrow \sum_{n=1}^{\infty} \frac{2n+5}{n^3+n+1}$ conv.

ex 4 $\sum_{n=1}^{\infty} \frac{(n^2+3)^{2/3}}{n\sqrt{n^2+1}}$

guess: $\frac{(n^2)^{2/3}}{n\sqrt{n^2}} = \frac{n^{4/3}}{n^2} = \frac{1}{n^{2/3}}$

$\sum \frac{1}{n^{2/3}}$ div.

$$a_n = \frac{(n^2+3)^{2/3}}{n\sqrt{n^2+1}} \qquad b_n = \frac{1}{n^{2/3}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{(n^2+3)^{2/3}}{n\sqrt{n^2+1}}}{\dfrac{1}{n^{2/3}}}$$

$$= \lim_{n\to\infty} \frac{n^{2/3}(n^2+3)^{2/3}}{n\sqrt{n^2+1}}$$

$$= \lim_{n\to\infty} \frac{n^{2/3} \cdot n^{4/3} \left(1+\frac{3}{n^2}\right)^{2/3}}{n \cdot n \cdot \sqrt{1+\frac{1}{n^2}}} = 1$$

limit comp: $\sum \frac{1}{n^{2/3}}$ div $\Rightarrow$

$\sum a_n$ div.