9.4 Infinite series

$\{a_n\}_{n=1}^{\infty}$ a sequence.

The notation $\sum_{n=1}^{\infty} a_n$

is called a <u>series</u>.

When $\lim_{j\to\infty} (a_1 + a_2 + \cdots + a_j)$

exists, then $\sum_{n=1}^{\infty} a_n$ <u>Converges</u>

and $\sum_{n=1}^{\infty} a_n = \lim_{j \to \infty} (a_1 + a_2 + \cdots + a_j)$.

When lim DNE, then $\sum_{n=1}^{\infty} a_n$ diverges.

$S_j = a_1 + a_2 + \cdots + a_j$ is called the j-th partial sum

For $\sum_{n=m}^{\infty} a_n$,

$$s_j = \underbrace{a_m + a_{m+1} + \cdots + a_{m+j-1}}_{j \text{ elements.}}$$

Formula:

$$1 + a + a^2 + \dots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

$$(a \neq 1)$$

<u>ex 1</u> $\sum_{n=0}^{\infty} \frac{1}{2^n}$ "geometric series"

$a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$,

$S_1 = 1$, $S_2 = 1 + \frac{1}{2}$,

$S_3 = 1 + \frac{1}{2} + \frac{1}{4}$

$$\lim_{j\to\infty} S_j = \lim_{j\to\infty}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{j-1}}\right)$$

$$= \lim_{j\to\infty} \frac{1-\left(\frac{1}{2}\right)^j}{1-\frac{1}{2}} = 2$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$$

<u>ex 2</u> $\sum_{n=1}^{\infty} 3^n$

$$S_j = a_1 + a_2 + \cdots + a_j$$

$$= 3 + 3^2 + \cdots + 3^j$$

$$\lim_{j\to\infty} S_j = \lim_{j\to\infty} (3 + 3^2 + \cdots + 3^j)$$

$$= \lim_{j\to\infty} 3\cdot(1 + 3 + 3^2 + \cdots + 3^{j-1})$$

$$= \lim_{j \to \infty} 3 \cdot \frac{1 - 3^j}{1 - 3} = \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} 3^n \text{ diverges.}$$

ex 3 $\sum_{n=m}^{\infty} r^n$

$$s_j = a_m + a_{m+1} + \cdots + a_{m+j-1}$$

$$= r^m + r^{m+1} + \cdots + r^{m+j-1}$$

$$\lim_{j\to\infty} s_j = \lim_{j\to\infty} \left(r^m + r^{m+1} + \cdots + r^{m+j-1}\right)$$

$$= \lim_{j\to\infty} r^m \left(1 + r + r^2 + \cdots + r^{j-1}\right)$$

$$= \lim_{j\to\infty} r^m \frac{1-r^j}{1-r} \qquad (r \neq 1)$$

$$|r| < 1 \qquad \lim_{j\to\infty} S_j = \frac{r^m}{1-r}$$

$$|r| > 1 \qquad \lim_{j\to\infty} S_j \text{ DNE}$$

$r = 1$, $\quad S_j = j$, $\lim_{j\to\infty} S_j$ DNE

$r = -1$, $\quad \lim_{j\to\infty} S_j$ DNE (oscillating).

Conclusion:

$$|r| < 1, \quad \sum_{n=m}^{\infty} r^n = \frac{r^m}{1-r}$$

$$|r| \geq 1, \quad \sum_{n=m}^{\infty} r^n \text{ diverges.}$$

Theorem

If $\sum a_n$ conv, then $\lim_{n\to\infty} a_n = 0$

If "$\lim_{n\to\infty} a_n = 0$" doesn't hold,

then $\sum a_n$ div.

ex 4 $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$a_n = \frac{n \cdot n \cdot \cdots \cdots n}{1 \cdot 2 \cdots \cdots n}$$

$$= \frac{n}{1} \cdot \frac{n}{2} \cdots \cdots \frac{n}{n} \geq 1$$

" $\displaystyle\lim_{n \to \infty} a_n = 0$ " doesn't hold

$\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ div.

ex 5 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$

"telescoping series"

$$\frac{1}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3}$$

$$1 = A\cdot(n+3) + B\cdot(n+1)$$

$n=-1$: $1 = A(-1+3)$ $A = \frac{1}{2}$

$n=-3$: $1 = B(-3+1)$ $B = -\frac{1}{2}$

$$\frac{1}{(n+1)(n+3)} = \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$

$$S_j = a_1 + a_2 + \cdots + a_j$$

$$= \frac{1}{2}\left[\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right)\right.$$

$$+ \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{6} - \frac{1}{8}\right)$$

$$\left. + \cdots + \left(\frac{1}{j} - \frac{1}{j+2}\right) + \left(\frac{1}{j+1} - \frac{1}{j+3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}+\frac{1}{3}-\frac{1}{j+2}-\frac{1}{j+3}\right].$$

$$\rightarrow \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)=\frac{5}{12}.$$

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+3)}=\frac{5}{12}.$$