SOLUTION

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name and your TA's name on every answer sheet.
(3) You can use any method to solve the problems (unless stated otherwise), but you have to justify your answers.
(4) You do not need to simplify your answer (you can leave $62.5 \cdot (10^6 - 7^6)\pi^2$ as final answer, but you cannot leave $\int_0^1 x\,\mathrm{d}x$ as final answer).
(5) You are not allowed to use calculators in this exam.

Problem 1. (30 points) Compute the following integrals:

(1) (15 points) $\int_0^\pi x\sin(2x)\,\mathrm{d}x$

$$\int_0^\pi x\sin(2x)\,\mathrm{d}x = x(-\frac{1}{2}\cos(2x))|_0^\pi - \int_0^\pi(-\frac{1}{2}\cos(2x)) = -\frac{\pi}{2} + \frac{1}{4}\sin(2x)|_0^\pi = -\frac{\pi}{2}$$

integration by parts: $u = x$, $\mathrm{d}v = \sin(2x)\,\mathrm{d}x$

(2) (15 points) $\int e^x \ln(1+e^x)\,\mathrm{d}x$

$$\int e^x\ln(1+e^x)\,\mathrm{d}x = \int \ln y\,\mathrm{d}y = y\ln y - \int \frac{1}{y}y\,\mathrm{d}y = y\ln y - y + C = (e^x+1)\ln(e^x+1) - (e^x+1) + C$$

first, substitution $y = 1 + e^x$; then integration by parts: $u = \ln y$, $\mathrm{d}v = \mathrm{d}y$. Notice that $e^x\ln(e^x+1) - e^x + C$ is also correct.

Problem 2. (20 points) Compute the following integral **by trigonometric substitution**: (your answer CANNOT contain the trig function of an inverse trig function: something like $\sin(2\cos^{-1}x)$.)

$$\int (1-4x^2)^{1/2}\,\mathrm{d}x$$

$$\int (1-4x^2)^{1/2}\,\mathrm{d}x = \int \sqrt{1 - 4\frac{1}{4}\sin^2 u} \cdot \frac{1}{2}\cos u\,\mathrm{d}u = \frac{1}{2}\int \cos^2 u\,\mathrm{d}u = \frac{1}{4}\int(1+\cos(2u))\,\mathrm{d}u$$

$$= \frac{1}{4}u + \frac{1}{8}\sin(2u) + C = \frac{1}{4}\sin^{-1}(2x) + \frac{1}{2}x\sqrt{1-4x^2} + C$$

Trig substitution: $x = \frac{1}{2}\sin u$, $\mathrm{d}x = \frac{1}{2}\cos u\,\mathrm{d}u$. Last step uses $u = \sin^{-1}(2x)$, $\sin(2u) = 2\sin u\cos u = 2 \cdot 2x \cdot \sqrt{1-4x^2}$

Problem 3. (30 points)

(1) (20 points) Compute the integral

$$\int \frac{x}{(x+2)(x^2+1)}\,\mathrm{d}x$$

Partial fractions:

$$\frac{x}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$
$$x = A(x^2+1) + (Bx+C)(x+2)$$

Letting $x=-2$ gives $A=-\frac{2}{5}$; comparing constant coefficient gives $0 = A+2C$, therefore $C=\frac{1}{5}$; comparing x^2 coefficient gives $0 = A+B$, therefore $B=\frac{2}{5}$.

$$\int \frac{x}{(x+2)(x^2+1)}\,\mathrm{d}x = \int \left(-\frac{2}{5}\frac{1}{x+2} + \frac{2}{5}\frac{x}{x^2+1} + \frac{1}{5}\frac{1}{x^2+1}\right)\mathrm{d}x$$

$$= -\frac{2}{5}\ln|x+2| + \frac{1}{5}\ln(x^2+1) + \frac{1}{5}\tan^{-1}x + C$$

where the second term uses the integral $\int \frac{x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\ln(x^2+1)+C$ which can be obtained by a substitution $u = x^2+1$.

(2) (10 points) Approximate the integral $\int_1^3 \sqrt{1+\frac{1}{x}}\,\mathrm{d}x$ by the trapezoidal rule with $n=4$. (You do NOT need to simplify the summation or estimate its error.)
The length of one small interval is $\frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$. Therefore x_0, x_1, x_2, x_3, x_4 are $1, \frac{3}{2}, 2, \frac{5}{2}, 3$. Corresponding function values $f(x_k)$ are $\sqrt{2}, \sqrt{\frac{5}{3}}, \sqrt{\frac{3}{2}}, \sqrt{\frac{7}{5}}, \sqrt{\frac{4}{3}}$. Therefore trapezoidal rule gives

$$\int_1^3 \sqrt{1+\frac{1}{x}}\,\mathrm{d}x \approx \frac{3-1}{2\cdot 4}[1\cdot\sqrt{2} + 2\cdot\sqrt{\frac{5}{3}} + 2\cdot\sqrt{\frac{3}{2}} + 2\cdot\sqrt{\frac{7}{5}} + 1\cdot\sqrt{\frac{4}{3}}]$$

Problem 4. (20 points) Determine the convergence/divergence of the following improper integral: (justify your answer!)

$$\int_0^\infty \frac{1}{x^2+x}\,\mathrm{d}x$$

Both 0 and ∞ are problematic points. Therefore

$$\int_0^\infty \frac{1}{x^2+x}\,\mathrm{d}x = \int_0^1 \frac{1}{x^2+x}\,\mathrm{d}x + \int_1^\infty \frac{1}{x^2+x}\,\mathrm{d}x = \lim_{c\to 0+}\int_c^1 \frac{1}{x^2+x}\,\mathrm{d}x + \lim_{c\to\infty}\int_1^c \frac{1}{x^2+x}\,\mathrm{d}x$$

First check the existence of the first limit. Compute (by partial fractions)

$$\int \frac{1}{x^2+x}\,\mathrm{d}x = \int (\frac{1}{x} - \frac{1}{x+1})\,\mathrm{d}x = \ln|x| - \ln|x+1| + C$$

$$\lim_{c\to 0+}\int_c^1 \frac{1}{x^2+x}\,\mathrm{d}x = \lim_{c\to 0+}(\ln|x| - \ln|x+1|)|_c^1 = \lim_{c\to 0+}((\ln 1 - \ln 2) - (\ln c - \ln(c+1))) = \infty$$

Last step is because $\ln c \to -\infty$, and all other terms goes to finite numbers. Therefore the limit $\lim_{c\to 0+} \int_c^1 \frac{1}{x^2+x}\,\mathrm{d}x$ DNE, and the original integral diverges.
Notice that we don't even need to check the second limit (which actually exists). If you get the second limit correctly, you will receive some partial credit.
The first integral $\int_0^1$ can also be shown to diverge by comparison theorem: use

$$\frac{1}{x^2+x} \geq \frac{1}{2x} \geq 0$$

for $0 < x \leq 1$, and the fact that $\int_0^1 \frac{1}{2x}\,\mathrm{d}x$ diverges. You could use the last fact directly, since its an improper integral of type $\frac{1}{x^p}$.