Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name and your TA's name on every answer sheet.
(3) You can use any method to solve the problems, but you have to justify your answers.
(4) You do not need to simplify your answer (you can leave $62.5 \cdot (10^6 - 7^6)\pi^2$ as final answer, but you cannot leave $\int_0^1 x\,\mathrm{d}x$ as final answer).
(5) You are not allowed to use calculators in this exam.

Problem 1. (25 points)

(1) (15 points) Find the largest interval containing $x = 2$ such that the function $f(x) = x^3 - 3x$ has an inverse. If we define $f(x)$ on this interval, what are the domain and range of $f^{-1}(x)$?

(2) (10 points) Find the inverse function of $g(x) = \frac{2}{1+e^{-x}}$. (Write your answer as $g^{-1}(x) = \cdots$)

(1) We want the monotone interval of f containing $x = 2$. $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$. $f'(x) = 0$ at $x = \pm 1. f'(x)$ is positive on $(-\infty, -1]$ and $[1, \infty)$. $f'(x)$ is negative on $[-1, 1]$. Therefore the monotone interval of f containing $x = 2$ is $[1, \infty)$.
The range of f^{-1} is the domain of f, which is $[1, \infty)$. The domain of f^{-1} is the range of f, for $x \in [1, \infty)$. Since f is increasing, we only need to compute the endpoint values. $f(1) = -2$, $\lim_{x\to\infty} f(x) = \infty$. Thus the domain of f^{-1} is $[-2, \infty)$.
(2)

$$y = \frac{2}{1+e^{-x}}, \quad 1 + e^{-x} = \frac{2}{y}, \quad e^{-x} = \frac{2}{y} - 1, \quad -x = \ln(\frac{2}{y} - 1), \quad x = -\ln(\frac{2}{y} - 1)$$

Thus

$$g^{-1}(x) = -\ln(\frac{2}{x} - 1)$$

Problem 2. (25 points) Compute the following derivative or integral:

(1) (5 points) $(2^{x^2+1})'$

(2) (10 points) $\int_1^9 \frac{\log_3 x}{x}\,\mathrm{d}x$

(3) (10 points) $\int \frac{e^x}{1+e^{2x}}\,\mathrm{d}x$

(1)

$$(2^{x^2+1})' = (e^{(x^2+1)\ln 2})' = e^{(x^2+1)\ln 2}(2\ln 2)x$$

(2)

$$\int \frac{\log_3 x}{x}\,\mathrm{d}x = \int \frac{\ln x}{(\ln 3)\cdot x}\,\mathrm{d}x = \frac{1}{\ln 3}\int \frac{\ln x}{x}\,\mathrm{d}x = \frac{1}{\ln 3}\int u\,\mathrm{d}u = \frac{1}{2\ln 3}u^2 + C = \frac{1}{2\ln 3}(\ln x)^2 + C$$

with the substitution $u = \ln x$. Therefore

$$\int_1^9 \frac{\log_3 x}{x}\,\mathrm{d}x = \frac{1}{2\ln 3}(\ln x)^2|_1^9 = \frac{1}{2\ln 3}(\ln 9)^2 = 2\ln 3$$

(the last step simplification is not necessary)

(3)

$$\int \frac{e^x}{1+e^{2x}}\,\mathrm{d}x = \int \frac{1}{1+u^2}\,\mathrm{d}u = \tan^{-1} u + C = \tan^{-1}(e^x) + C$$

with the substitution $u = e^x$.

Problem 3. (25 points) Compute the following quantity or integral: (your answer for (1) cannot contain trig or inverse trig functions)

(1) (10 points) $\sin(\tan^{-1}(3))$

(2) (15 points) $\int \frac{1}{y^2+2y+3}\,\mathrm{d}y$

(1) $\tan^{-1}(3)$ is some angle between 0 and $\pi/2$ with tan value equal to 3, in other words, opposite / adjacent = 3 / 1 inside a right triangle. The hypotenuse is therefore $\sqrt{1^2+3^2} = \sqrt{10}$. Thus sin of that angle is opposite / hypotenuse which is $\frac{3}{\sqrt{10}}$.

(2)

$$\int \frac{1}{y^2+2y+3}\,\mathrm{d}y = \int \frac{1}{(y+1)^2+2}\,\mathrm{d}y = \frac{1}{2}\int \frac{1}{\frac{1}{2}(y+1)^2+1}\,\mathrm{d}y$$

$$= \frac{1}{2}\sqrt{2}\int \frac{1}{u^2+1}\,\mathrm{d}u = \frac{1}{\sqrt{2}}\tan^{-1} u + C = \frac{1}{\sqrt{2}}\tan^{-1}(\frac{y+1}{\sqrt{2}}) + C$$

with the substitution $u = \frac{y+1}{\sqrt{2}}$. (it is also ok if you use the formula $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$, and jump from $\int \frac{1}{(y+1)^2+2}\,\mathrm{d}y$ to the answer directly.)

Problem 4. (25 points) Compute the following limits:

(1) (5 points) $\lim_{x\to 0} \frac{x}{e^x}$

(2) (10 points) $\lim_{x\to 0}(\frac{\sin x}{x^3} - \frac{1}{x^2})$

(3) (10 points) $\lim_{x\to 0+} (1-\sin x)^{\frac{1}{x}}$

(1) By substituting $x = 0$ into the function, we get the limit equal to 0.

(2)

$$\lim_{x\to 0}(\frac{\sin x}{x^3} - \frac{1}{x^2}) = \lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{\cos x - 1}{3x^2} = \lim_{x\to 0}\frac{-\sin x}{6x} = \lim_{x\to 0}\frac{-\cos x}{6} = -\frac{1}{6}$$

where we used 0/0 type l'Hopital's rule three times.

(3)

$$\lim_{x\to 0+}(1-\sin x)^{\frac{1}{x}} = \lim_{x\to 0+} e^{\frac{1}{x}\ln(1-\sin x)} = e^{\lim_{x\to 0+}\frac{\ln(1-\sin x)}{x}} = e^{\lim_{x\to 0+}\frac{\frac{1}{1-\sin x}(-\cos x)}{1}} = e^{-1}$$

where we used 0/0 type l'Hopital's rule for the limit on the exponent.