9.3 Convergence of sequence

Theorem (squeezing)

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, and

$a_n \le c_n \le b_n$, then $\{c_n\}$

converges, and $\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n$.

ex 1 $\lim\limits_{n\to\infty} \dfrac{\sin(n^2)}{2^n}$

$$-1 \le \sin(n^2) \le 1$$

$$-\frac{1}{2^n} \le \frac{\sin(n^2)}{2^n} \le \frac{1}{2^n}$$

$$\lim_{n\to\infty}\left(-\frac{1}{2^n}\right) = 0 \qquad \lim_{n\to\infty} \frac{1}{2^n} = 0$$

$$\xRightarrow{\text{squeezing}} \lim_{n\to\infty} \frac{\sin(n^2)}{2^n} = 0.$$

__ex 2__ $\displaystyle \lim_{n\to\infty} n \sin\left(\frac{1}{n}\right)$

$$= \lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\underset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = 1$$

ex 3 $\displaystyle\lim_{n\to\infty} \frac{n}{2n+\sin n}$

$-1 \le \sin n \le 1$ $(n \ge 1)$

$$\frac{n}{2n+1} \le \frac{n}{2n+\sin n} \le \frac{n}{2n-1}$$

$\displaystyle\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$ $\displaystyle\lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}$

squeezing $\Longrightarrow$ $\displaystyle\lim_{n\to\infty} \frac{n}{2n+\sin n} = \frac{1}{2}$

ex 4 $\lim_{n\to\infty}\left(\sqrt{n+2}-\sqrt{n}\right)$

$$= \lim_{n\to\infty} \frac{\left(\sqrt{n+2}-\sqrt{n}\right)\left(\sqrt{n+2}+\sqrt{n}\right)}{\sqrt{n+2}+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{(n+2)-n}{\sqrt{n+2}+\sqrt{n}} = 0$$

$\{a_n\}$ is **bounded** if there exists M such that $|a_n| \leq M$ for all n.

Theorem

$\{a_n\}$ converges $\Rightarrow$ $\{a_n\}$ bounded

$\{a_n\}$ unbounded $\Rightarrow$ $\{a_n\}$ diverges

<u>exs</u>: bounded or not:

$\{(-1)^n\}$	✓	div
$\left\{\dfrac{(-1)^n}{n}\right\}$	✓	conv
$\{n(-1)^n\}$	✗	div
$\{\sin n\}$	✓	

$\{a_n\}$ is <u>increasing</u> if $a_{n+1} > a_n$ for all n

<u>decreasing</u> if $a_{n+1} < a_n$ for all n

<u>Theorem</u> If $\{a_n\}$ bounded, and either inc. or dec., then $\{a_n\}$ converges.

ex 6 Define $\{a_n\}$ by

$a_0 = 0$, $a_{n+1} = a_n^2 + \frac{1}{4}$

"recursively defined"

Show: 1 $\{a_n\}$ bounded

2 $\{a_n\}$ increasing

$\Rightarrow$ $\{a_n\}$ converges

① To show "bounded"

Want to show by induction that

$$|a_n| < \frac{1}{2}.$$

(1) true for $n = 0$

(2) Suppose $|a_n| < \frac{1}{2}$, then

$$a_{n+1} = a_n^2 + \frac{1}{4}$$

$$|a_{n+1}| \leq |a_n|^2 + \frac{1}{4} < \left(\frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{2}$$

✓

2 To show $\{a_n\}$ increasing.

$$a_{n+1} - a_n = a_n^2 + \frac{1}{4} - a_n$$

$$= \left(a_n - \frac{1}{2}\right)^2 > 0$$

because $|a_n| < \frac{1}{2}$
by 1

Since $\lim_{n \to \infty} a_n$ exists,

$$\lim_{n \to \infty} a_{n+1} = \left(\lim_{n \to \infty} a_n\right)^2 + \frac{1}{4}$$

$$\lim_{n \to \infty} a_n = \left(\lim_{n \to \infty} a_n\right)^2 + \frac{1}{4}$$

$$\Rightarrow \lim_{n \to \infty} a_n = \frac{1}{2}$$