9.2 Sequence

A **sequence**: $a_1, a_2, a_3, \cdots, a_n, \cdots$

"a function whose domain is $\{1, 2, 3, \cdots\}$".

Notation: $\{a_n\}_{n=1}^{\infty}$

ex 1 Write the first 3 terms of

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} : \underset{a_1}{\underset{\uparrow}{1}}, \underset{a_2}{\underset{\uparrow}{\frac{1}{2}}}, \underset{a_3}{\underset{\uparrow}{\frac{1}{3}}}, \ldots$$

$$\left\{(-1)^n\right\}_{n=0}^{\infty} : \underset{a_0}{\underset{\uparrow}{1}}, \underset{a_1}{\underset{\uparrow}{-1}}, \underset{a_2}{\underset{\uparrow}{1}}, \ldots$$

The <u>limit</u> of a sequence $\{a_n\}_{n=1}^{\infty}$ is a number L such that $|a_n - L|$ gets arbitrarily small if n gets large.

Notation: $\lim\limits_{n\to\infty} a_n = L$

(limit may exist or DNE)

$\lim_{n \to \infty} a_n = \infty$ means a_n can gets arbitrarily large, if n gets large.

(Similarly define $\lim_{n \to \infty} a_n = -\infty$)

Both are cases of "DNE"

Theorem If a function $f(x)$ satisfies $f(n) = a_n$ for all n, and $\lim_{x \to \infty} f(x) = L$ or ∞ or $-\infty$, then $\lim_{n \to \infty} a_n$ is the same.

ex 2 $\lim_{n \to \infty} \frac{1}{n} = \lim_{x \to \infty} \frac{1}{x} = 0$

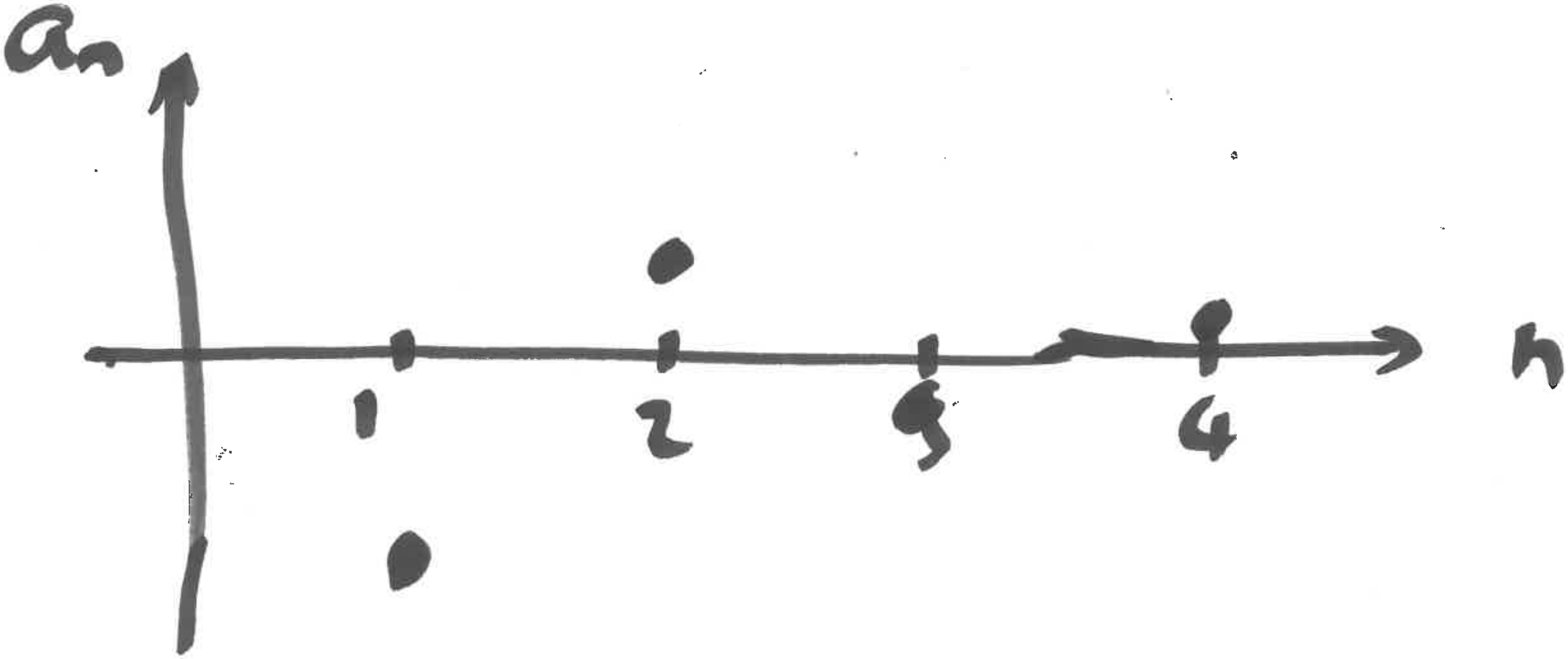
$$\lim_{n \to \infty} \frac{n^2}{n^2+1} = \lim_{x \to \infty} \frac{x^2}{x^2+1} = 1$$

$$\lim_{n \to \infty} \frac{e^n}{n} = \lim_{x \to \infty} \underbrace{\frac{e^x}{x}}_{\text{"}\frac{\infty}{\infty}\text{"}} = \lim_{x \to \infty} \frac{e^x}{1}$$

$$= \infty$$

$$\lim_{n\to\infty} \frac{(-1)^n}{n^2+1}$$

$$\frac{-1}{2}, \frac{1}{5}, \frac{-1}{10}, \frac{1}{17}, \ldots$$



$$\lim_{n\to\infty} \frac{1}{n^2+1} = \lim_{x\to\infty} \frac{1}{x^2+1} = 0$$

$$\left| \frac{(-1)^n}{n^2+1} - 0 \right| = \left| \frac{(-1)^n}{n^2+1} \right|$$

$$= \frac{1}{n^2+1} \to 0 \quad (n\to\infty)$$

$$\Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{-n}$$

$$= \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{-x}$$

$$= \lim_{x \to \infty} e^{-x \cdot \ln\left(1 + \frac{1}{x}\right)}$$

see next page

$$= e^{\lim_{x \to \infty} -x \ln\left(1 + \frac{1}{x}\right)} = \boxed{e^{-1}}$$

$$\lim_{x\to\infty} -x \ln\left(1+\frac{1}{x}\right)$$

$$= -\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\underset{\text{"}\frac{0}{0}\text{"}}{\overset{\uparrow}{=}} -\lim_{x\to\infty} \frac{\frac{1}{1+1/x}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= -\lim_{x\to\infty} \frac{1}{1+1/x} = -1$$

$$\lim_{n\to\infty} \cos(2\pi n) = \lim_{x\to\infty} \cos(2\pi x)$$

DNE, not $\pm\infty$

$$\{\cos(2\pi n)\}_{n=1}^{\infty}$$

$$1,\ 1,\ 1,\ \ldots$$

$$\lim_{n\to\infty} \cos(2\pi n) = 1$$

$$\lim_{n\to\infty} \frac{2^n}{n!} = \lim_{n\to\infty} \frac{2\cdot 2\cdot 2\cdot 2\cdot\cdots\cdot 2\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdots (n-1)n}$$

$$= \lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdot\underbrace{\frac{2}{4}\cdot\frac{2}{5}\cdot\frac{2}{6}\cdots\cdots\frac{2}{n-1}\cdot\frac{2}{n}}_{\substack{\text{every factor} \\ \text{is } \leq \frac{1}{2} \\ (n-3)\text{ factors}}}$$

$$a_n \leq \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \cdot \left(\frac{1}{2}\right)^{n-3} \longrightarrow 0$$

$$\Rightarrow \lim_{n \to \infty} a_n = 0.$$