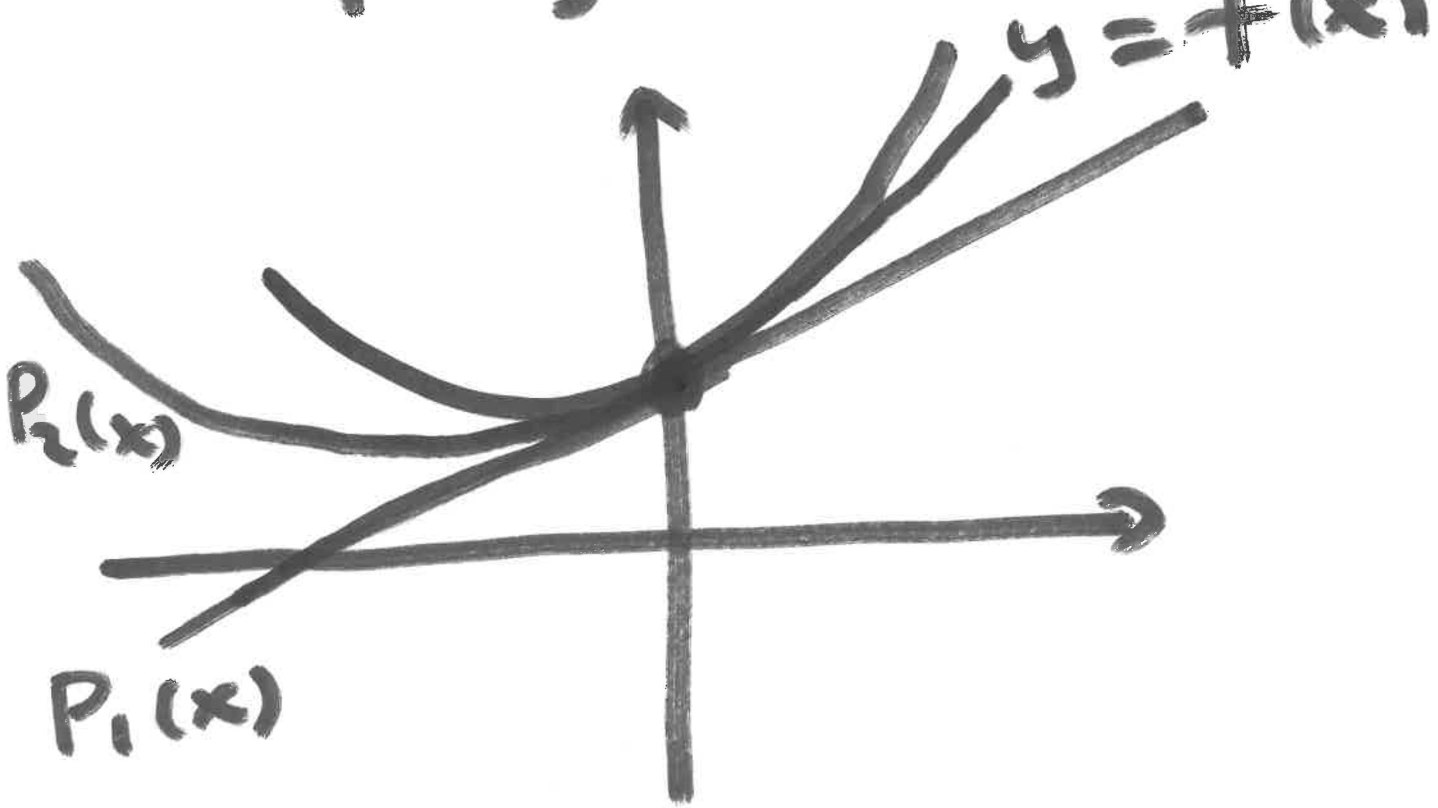
9.1 Polynomial approx.

Given $f(x)$. The <u>n-th order Taylor polynomial</u>

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

It is the best approx. of $f(x)$ near $x=0$ by degree n polynomial.



To compute $P_n(x)$ for $f(x)$:
take n-th derivative of f.

evaluate derivatives at $x=0$.

. If $f(x)$ is a polynomial,
then $P_n(x)$ is the poly.
containing all terms in f
with degree $\leq n$.

ex 1 $f(x) = x^2 + 2x - 3$

$P_1(x) = -3 + 2x$

$P_{10}(x) = -3 + 2x + x^2$

ex 2

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad \vdots$

$\vdots$

$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$$

$$+ \cdots + \frac{1}{n!}x^n$$

<u>ex 3</u> $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f^{(3)}(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$\vdots$

Want $P_{4n}(x)$

$f(0) = 0$

$f'(0) = 1$

$f''(0) = 0$

$f^{(3)}(0) = -1$

$f^{(4)}(0) = 0$

$f^{(4n-3)}(x) = \cos x$ $\quad 1$

$f^{(4n-2)}(x) = -\sin x$ $\quad 0$

$f^{(4n-1)}(x) = -\cos x$ $\quad -1$

$f^{(4n)}(x) = \sin x$ $\quad 0$

$$P_{4n}(x) = 0 + 1 \cdot x + \frac{0}{2!}x^2$$

$$+ \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \cdots$$

$$+ \frac{1}{(4n-3)!}x^{4n-3} + \frac{0}{(4n-2)!}x^{4n-2}$$

$$+ \frac{-1}{(4n-1)!}x^{4n-1} + \frac{0}{(4n)!}x^{4n}$$

$$P_{4n} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$

$$+ \cdots + \frac{1}{(4n-3)!}x^{4n-3} - \frac{1}{(4n-1)!}x^{4n-1}$$

ex 4 $\quad f(x) = \dfrac{1}{1-x}$

$$f'(x) = (-1)\frac{1}{(1-x)^2}\cdot(-1) = \frac{1}{(1-x)^2}$$

$$f''(x) = (-2)\cdot\frac{1}{(1-x)^3}\cdot(-1) = \frac{2}{(1-x)^3}$$

$$f^{(3)}(x) = (-3)\frac{2}{(1-x)^4}\cdot(-1) = \frac{3\cdot 2}{(1-x)^4}$$

$$\vdots$$

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

$$f(0) = 1$$

$$f'(0) = 1$$

$$f''(0) = 2!$$

$$f^{(3)}(0) = 3!$$

$$f^{(n)}(0) = n!$$

$$P_n(x) = 1 + 1\cdot x + \frac{2!}{2!}x^2 + \frac{3!}{3!}x^3 + \cdots + \frac{n!}{n!}x^n$$

$$= 1 + x + x^2 + \cdots + x^n$$

ex 5

$$f(x) = \frac{x^2+1}{1-x}$$

compute $P_6(x)$.

$$\begin{array}{r|l} & -x-1 \\ \hline -x+1 & x^2+0\cdot x+1 \\ & x^2-x \\ \hline & x+1 \\ & x-1 \\ \hline & 2 \end{array}$$

$$f(x) = \underbrace{-x-1} + \underbrace{\frac{2}{1-x}}$$

$$\text{"}P_6(x)\text{"} = -x-1$$

"$P_6(x)$" for $\dfrac{2}{1-x}$ is

$$2(1 + x + x^2 + \cdots + x^6)$$

"$P_6(x)$" for $f(x)$ is

$$-x-1 + 2(1 + x + x^2 + \cdots + x^6)$$

ex 6 $\quad f(x) = e^{-x^2}$. Compute P_4

$$f'(x) = e^{-x^2} \cdot (-2x)$$

$$f''(x) = e^{-x^2} \cdot (-2x) \cdot (-2x)$$

$$+ e^{-x^2} \cdot (-2)$$

$$= e^{-x^2} \cdot (4x^2 - 2)$$

$$f^{(3)}(x) = (-8x^3 + 12x)e^{-x^2}$$

$$f^{(4)}(x) = (16x^4 - 48x^2 + 12)e^{-x^2}$$

$f(0) = 1$

$f'(0) = 0$

$f''(0) = -2$

$f^{(3)}(0) = 0$

$f^{(4)}(0) = 12$

$$P_4(x) = 1 + \frac{-2}{2!}x^2 + \frac{12}{4!}x^4$$

$$= 1 - x^2 + \frac{1}{2} x^4$$

Recall $P_2(x)$ for e^x is

$$1 + x + \frac{1}{2} x^2$$