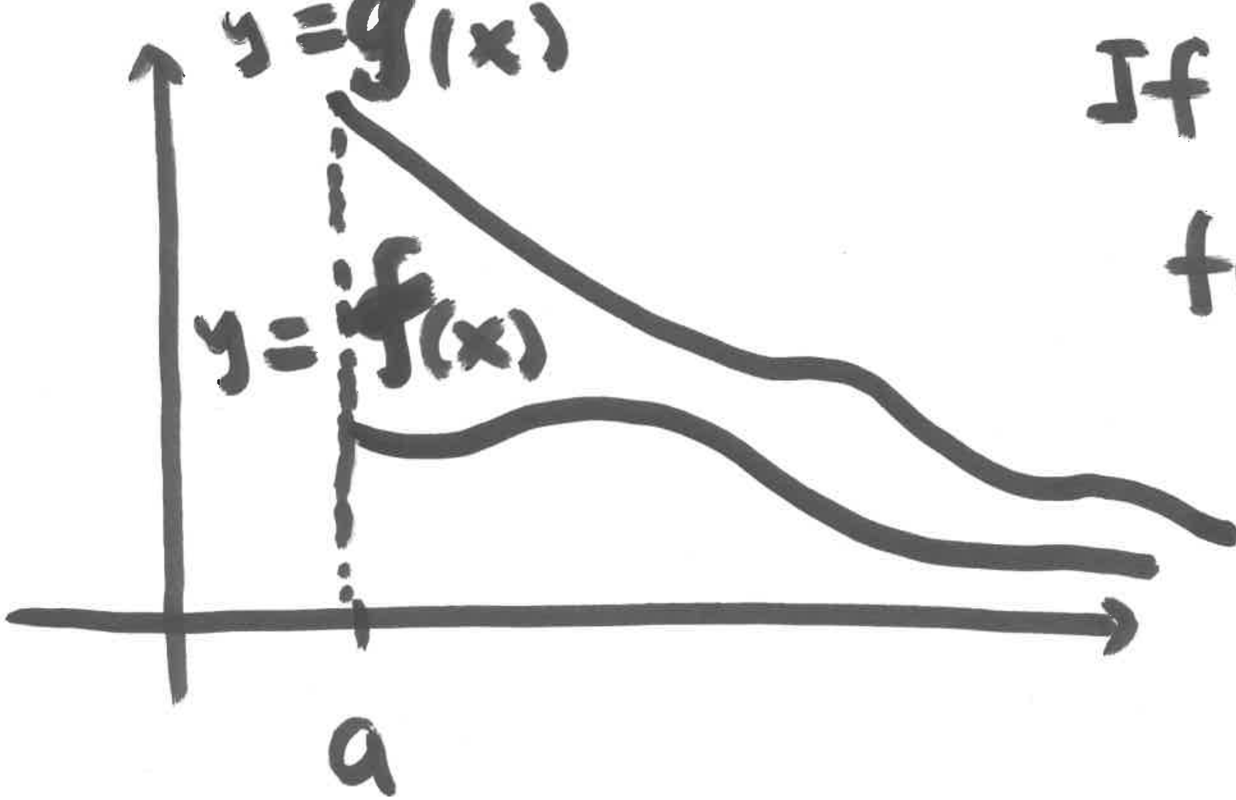
Comparison theorem



If $0 \le f(x) \le g(x)$ for $x \ge a$

- $\int_a^\infty g(x)\,dx$ conv $\Rightarrow \int_a^\infty f(x)\,dx$ conv
- $\int_a^\infty f(x)\,dx$ div $\Rightarrow \int_a^\infty g(x)\,dx$ div

- Guess conv/div
- Find the function to compare. (easier function)
- Adjust non-problematic endpoint.

ex 1 $\int_0^{\boxed{\infty}} e^{-x^2}\,dx$

want $e^{-x^2} \leq e^{-x}$

$$\Updownarrow$$

$$-x^2 \leq -x$$

$$\Updownarrow$$

$$x^2 - x \geq 0$$

$$\Updownarrow$$

$$x(x-1) \geq 0$$

true for $x \geq 1$

$$\int_0^{\boxed{\infty}} e^{-x^2}\,dx = \underbrace{\int_0^1 e^{-x^2}\,dx} \checkmark$$

$$+ \int_1^{\boxed{\infty}} e^{-x^2}\,dx$$

$$0 \le e^{-x^2} \le e^{-x} \quad \text{for } x \ge 1$$

$$\int_1^{\infty} e^{-x}\,dx = -e^{-x}\Big|_1^{\infty}$$

$$= -(0 - e^{-1}) = e^{-1} \quad \text{conv.}$$

By comparison, $\int_1^{\infty} e^{-x^2}\,dx$ conv.

$\int_0^\infty e^{-x^2}\,dx \quad \boxed{\text{conv.}}$

ex 2 $\displaystyle\int_0^{\boxed{\infty}} \frac{1}{1+x^{4/5}}\,dx$

Guess : div

Want: $\displaystyle \frac{1}{1+x^{4/5}} \geq \frac{1}{2x^{4/5}}$

$$2x^{4/5} \geq 1 + x^{4/5}$$

$x^{4/5} \geq 1$ true for $\underline{x \geq 1}$

$$= \underbrace{\int_0^1 \frac{1}{1+x^{4/5}}\,dx}_{\checkmark} + \int_1^{\boxed{\infty}} \frac{1}{1+x^{4/5}}\,dx$$

$$\frac{1}{1+x^{4/5}} \geq \frac{1}{2x^{4/5}} \geq 0$$

$$\int_1^\infty \frac{1}{2x^{4/5}}\,dx \quad \text{div}$$

By comparison. $\int_1^\infty \frac{1}{1+x^{4/5}}\,dx$ div

$$\Rightarrow \int_0^\infty \frac{1}{1+x^{4/5}}\,dx \quad \boxed{\text{div.}}$$

ex 3 $\displaystyle\int_0^3 \frac{1}{x^2-1}\,dx$

$$= \int_0^{\boxed{1}} \frac{1}{x^2-1}\,dx + \int_{\boxed{1}}^3 \frac{1}{x^2-1}\,dx$$

$$\int \frac{1}{x^2-1}\,dx = \int \frac{1}{(x+1)(x-1)}\,dx$$

$$= \int \left(\frac{A}{x+1} + \frac{B}{x-1} \right) dx$$

$$1 = A(x-1) + B(x+1)$$

$$x = 1 \Rightarrow 1 = B\cdot(1+1) \quad B = \frac{1}{2}$$

$$x = -1 \Rightarrow 1 = A(-1-1) \quad A = -\frac{1}{2}$$

$$\int \frac{1}{x^2-1}\,dx = \int \left(\frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1} \right) dx$$

$$= -\frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + C.$$

$$\lim_{c \to 1^-} \int_0^c \frac{1}{x^2-1}\,dx$$

$$= \lim_{c \to 1^-} \left(-\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| \right) \Big|_0^c$$

$$= \lim_{c \to 1^-} \Big(\underbrace{-\frac{1}{2}\ln(c+1)}_{\checkmark} + \underbrace{\frac{1}{2}\ln(1-c)}_{\to -\infty} \Big)$$

$$= -\infty \implies \text{div.}$$

$$\implies \int_0^3 \frac{1}{x^2-1}\,dx \quad \boxed{\text{div.}}$$

ex 4

$$\int_{\boxed{1}}^{\boxed{\infty}} \frac{1}{x\sqrt{\ln x}}\,dx$$

$$= \int_{\boxed{1}}^{2} \frac{1}{x\sqrt{\ln x}}\,dx + \int_{2}^{\boxed{\infty}} \frac{1}{x\sqrt{\ln x}}\,dx$$

$$\int \frac{1}{x\sqrt{\ln x}}\,dx = \int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} + C$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= 2\sqrt{\ln x} + C$$

① $\lim\limits_{c \to 1^+} \int_c^2 \frac{1}{x\sqrt{\ln x}}\,dx$

$= \lim\limits_{c \to 1^+} 2\sqrt{\ln x}\,\Big|_c^2$

$= \lim\limits_{c \to 1^+} \left(2\sqrt{\ln 2} - 2\sqrt{\ln c}\right)$

$= 2\sqrt{\ln 2} \Rightarrow \int_{\boxed{1}}^2$ conv.

② $$\lim_{c\to\infty} \int_2^c \frac{1}{x\sqrt{\ln x}}\,dx$$

$$= \lim_{c\to\infty} 2\sqrt{\ln x}\,\Big|_2^c$$

$$= \lim_{c\to\infty} \Big(\underbrace{2\sqrt{\ln c}}_{\to\infty} - 2\sqrt{\ln 2}\Big)$$

$$= \infty \qquad \Rightarrow \qquad \int_2^{\boxed{\infty}} \quad \text{div.}$$

$$\Rightarrow \int_{\boxed{1}}^{\boxed{\infty}} \quad \boxed{\text{div.}}$$