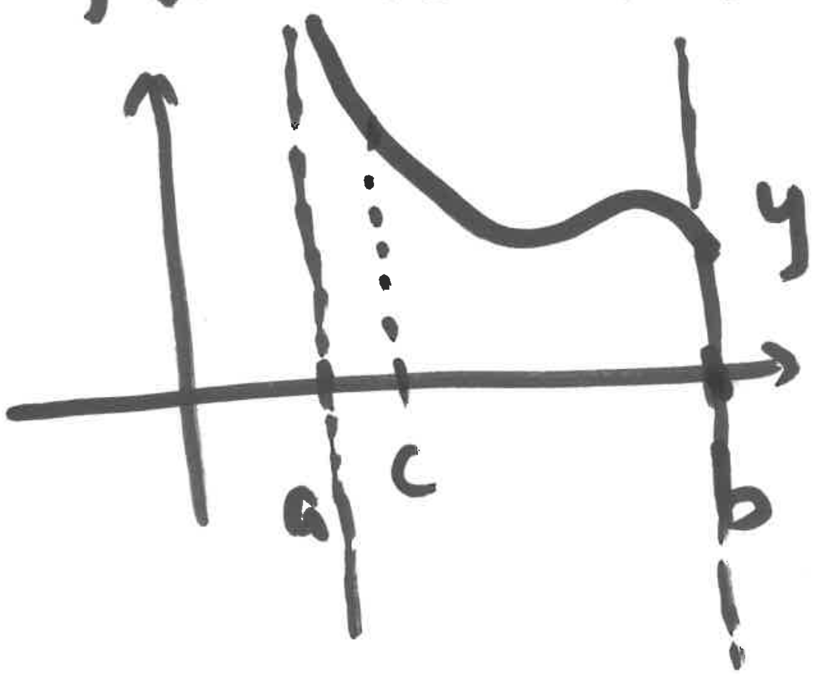
8.7 Improper integral

① $\int_a^b f(x)\,dx = \lim_{c \to a+} \int_c^b f(x)\,dx$

f(x) is not bounded on [a, b]



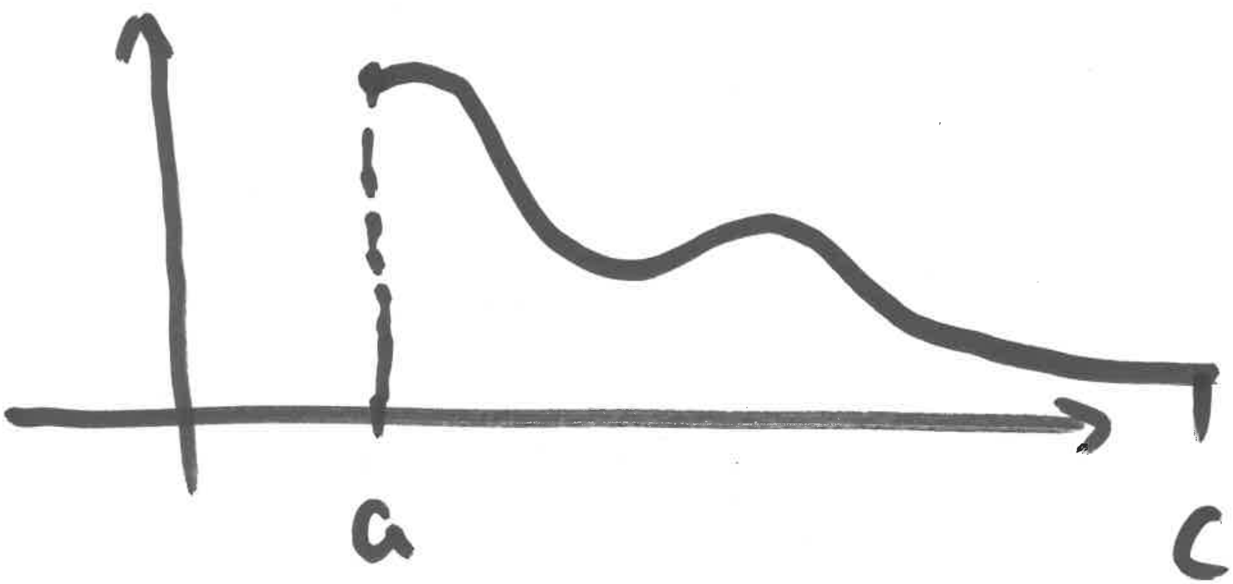
(for example, not bounded near x=a)

If limit exists, we say the improper integral $\int_a^b f(x)\,dx$ <u>converges</u>, otherwise it <u>diverges</u>.

② $\int_a^\infty f(x)\,dx = \lim_{c \to \infty} \int_a^c f(x)\,dx$

↑

interval is infinite



To determine an improper integral converges/diverges:

- Identify the problematic integral bound(s)
- If you can compute indef. integral, then evaluate limit explicitly.

ex 1 $\int_1^{\boxed{\infty}} \frac{1}{x^p}\,dx \quad (p>0)$

$$= \lim_{c\to\infty} \int_1^c \frac{1}{x^p}\,dx$$

① $p \neq 1$

$$\int_1^c \frac{1}{x^p}\,dx = \frac{1}{1-p}\cdot\frac{1}{x^{p-1}}\Big|_1^c$$

$$= \frac{1}{1-p}\left(\frac{1}{c^{p-1}} - 1\right)$$

$p > 1$

$$\lim_{c\to\infty} \frac{1}{1-p}\left(\frac{1}{c^{p-1}} - 1\right) = \frac{1}{p-1}$$

(exists).

$p < 1$

$$\lim_{c\to\infty} \frac{1}{1-p}\left(\boxed{\frac{1}{c^{p-1}}} - 1\right) = \infty$$

(DNE)

$= c^{1-p}$

② $p = 1$

$$\int_1^c \frac{1}{x}\,dx = \ln|x|\Big|_1^c = \ln c$$

$$\lim_{c\to\infty} \ln c = \infty \quad (\text{DNE})$$

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answer: $p > 1 \Rightarrow$ converges

$p \leq 1 \Rightarrow$ diverges

__ex 2__ $\displaystyle\int_{\boxed{0}}^{1} \frac{1}{x^p}\,dx \qquad (p>0)$

$$= \lim_{c\to 0+} \int_c^1 \frac{1}{x^p}\,dx$$

① $p \neq 1$

$$\int_c^1 \frac{1}{x^p}\,dx = \frac{1}{1-p} \cdot \frac{1}{x^{p-1}} \Big|_c^1$$

$$= \frac{1}{1-p}\left(1 - \frac{1}{c^{p-1}}\right)$$

$p > 1 \quad \lim_{c \to 0+} \frac{1}{1-p}\left(1 - \frac{1}{c^{p-1}}\right) = \infty$

(DNE)

$p < 1 \quad \lim_{c \to 0+} \frac{1}{1-p}\left(1 - \frac{1}{c^{p-1}}\right) = \frac{1}{1-p}$

$\hookrightarrow = c^{1-p}$ (exists)

② $p = 1 \quad \int_c^1 \frac{1}{x}\,dx = \ln|x| \Big|_c^1 = -\ln c$

$\lim_{c \to 0+} (-\ln c) = \infty$ (DNE) answer:

$p < 1 \Rightarrow$ converges

$p \geq 1 \Rightarrow$ diverges

	$0 < p < 1$	$p = 1$	$p > 1$
$\int_1^\infty \frac{1}{x^p}\,dx$	div	div	conv
$\int_0^1 \frac{1}{x^p}\,dx$	conv	div	div

ex 3

$$\int_{\boxed{-\infty}}^{\boxed{\infty}} x e^{-x^2} dx$$

$$\text{"}=\text{"} \int_0^{\boxed{\infty}} x e^{-x^2} dx + \int_{\boxed{-\infty}}^{0} x e^{-x^2} dx$$

$$= \lim_{c \to \infty} \int_0^c x e^{-x^2} dx$$

$$+ \lim_{c \to -\infty} \int_c^0 x e^{-x^2} dx$$

$$\int x e^{-x^2} dx = -\frac{1}{2} \int e^u du$$

$$\left(\begin{array}{l} u = -x^2 \\ du = -2x\,dx \end{array} \right)$$

$$= -\frac{1}{2} e^u + C$$

$$= -\frac{1}{2} e^{-x^2} + C$$

Say $\int_{-\infty}^{\infty}$ converges when

BOTH limits exist.

Otherwise it diverges.

① $\int_0^c x e^{-x^2} dx$

$= -\frac{1}{2} e^{-x^2} \Big|_0^c$

$= -\frac{1}{2}(e^{-c^2} - 1)$

$\lim_{c \to \infty} -\frac{1}{2}(e^{-c^2} - 1) = \frac{1}{2}$ (exists)

② $\int_c^0 x e^{-x^2}\,dx$

$= -\frac{1}{2} e^{-x^2} \Big|_c^0$

$= -\frac{1}{2}\left(1 - e^{-c^2}\right)$

$\lim_{c \to -\infty} -\frac{1}{2}\left(1 - e^{-c^2}\right) = -\frac{1}{2}$

(exists)

Conclusion: $\int_{-\infty}^{\infty} x e^{-x^2}\,dx$ conv.

$$\int_{-\infty}^{\infty} x e^{-x^2}\,dx = \underset{\int_0^{\infty}}{\underset{\uparrow}{\frac{1}{2}}} + \left(-\underset{\int_{-\infty}^{0}}{\underset{\uparrow}{\frac{1}{2}}}\right) = 0$$