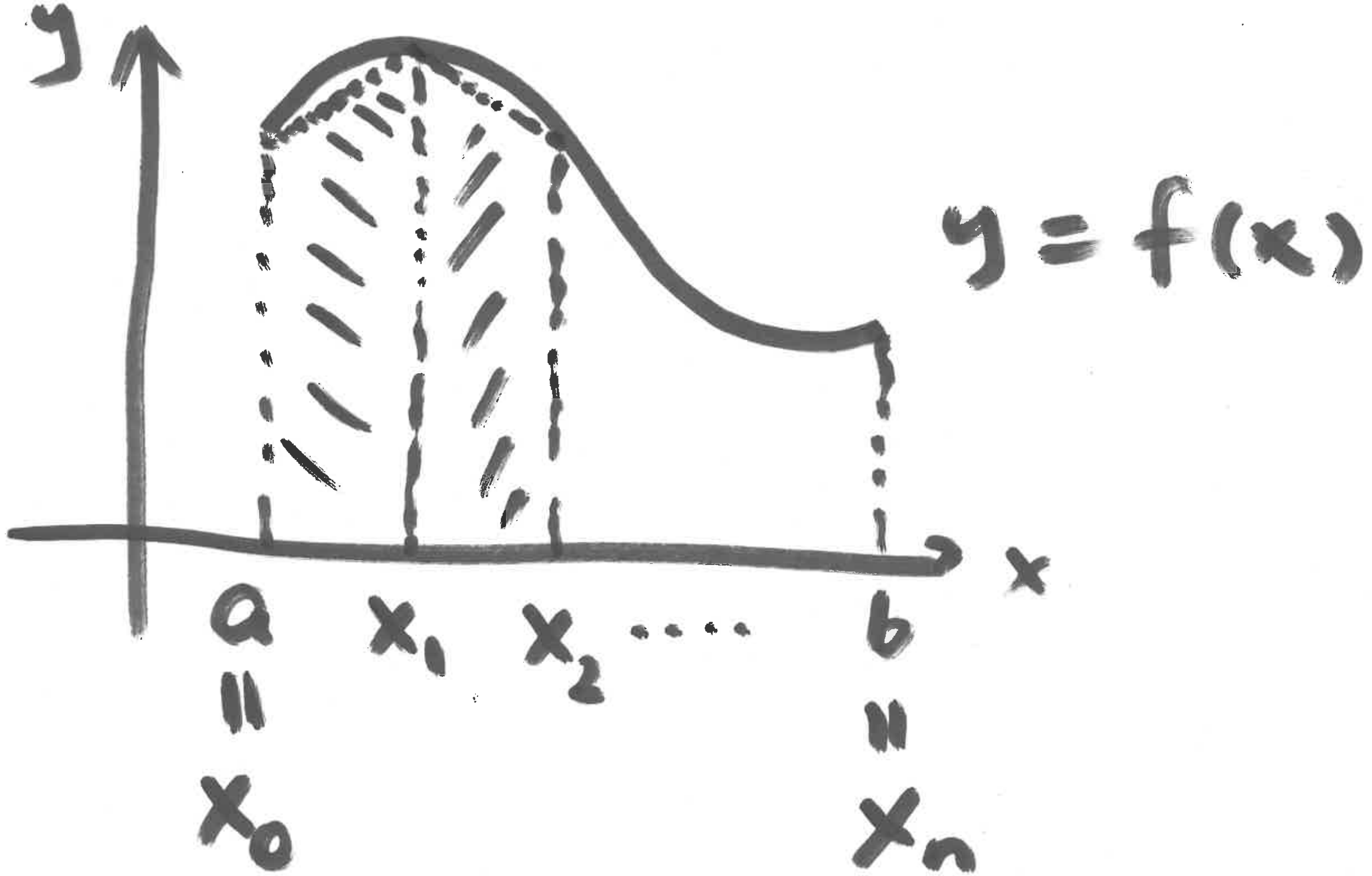
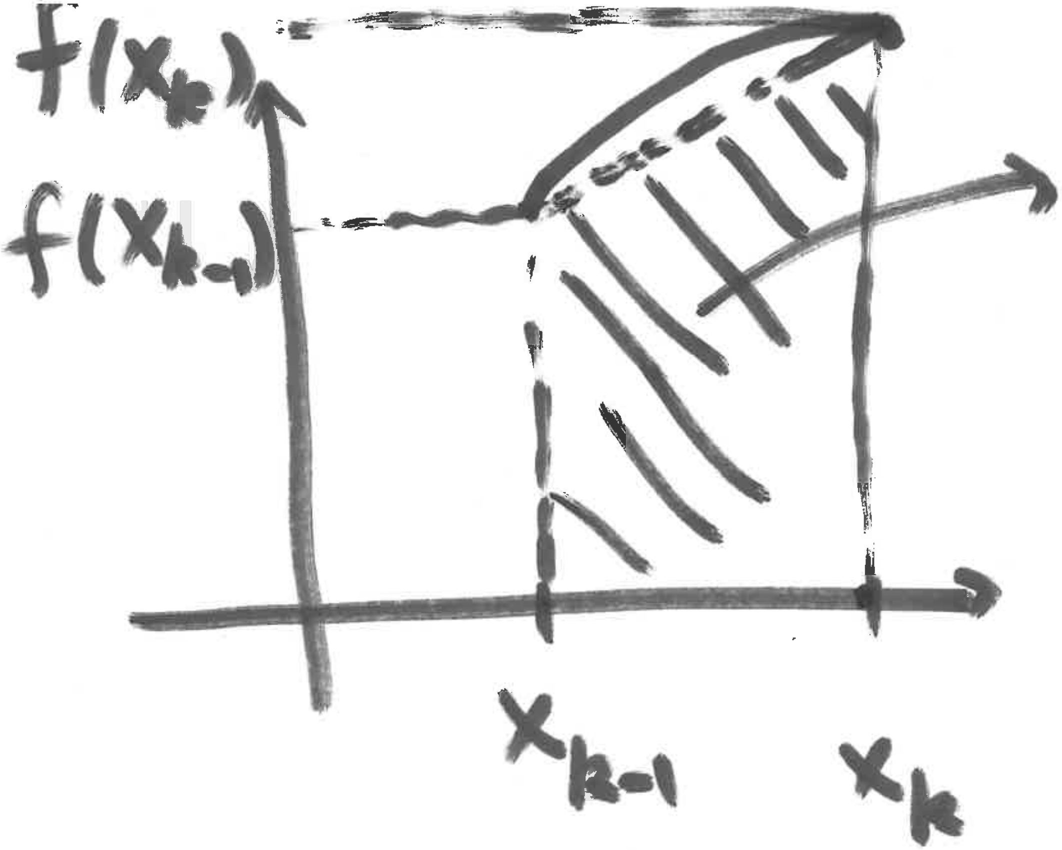
8.6 Trapezoidal rule, Simpson's rule

Approximately compute definite integral

$$\int_a^b f(x)\,dx$$


y
y = f(x)
x
a
x_1
x_2 ....
b
=
x_0
=
x_n


approx



area of trapezoid

$$= \frac{1}{2}\left(f(x_{k-1}) + f(x_k)\right) \cdot \frac{b-a}{n}$$

$$x_k - x_{k-1} = \frac{b-a}{n}$$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\Big[(f(x_0)+f(x_1)) + (f(x_1)+f(x_2)) + \cdots + (f(x_{n-1})+f(x_n))\Big]$$

$$= \frac{b-a}{2n}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big]$$

trapezoidal rule

$$E_n^T = \left| \int_a^b f(x)\,dx - \frac{b-a}{2n}[\cdots\cdots] \right|$$

$$E_n^T \le \frac{\max_{a \le x \le b} |f''(x)|}{12n^2} (b-a)^3$$

ex 1 Approximate $\int_1^2 \frac{1}{x}\,dx$ (n=4)

by trape. rule. Estimate error

(exact answer = $\ln 2 = 0.693147\cdots$)

k	0	1	2	3	4
x_k	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2
$f(x_k)$	1	$\frac{4}{5}$	$\frac{2}{3}$	$\frac{4}{7}$	$\frac{1}{2}$

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{2-1}{2\cdot 4}\left[1+2\cdot\frac{4}{5}+2\cdot\frac{2}{3}\right.$$

$$\left.+2\cdot\frac{4}{7}+\frac{1}{2}\right]=0.697023\cdots$$

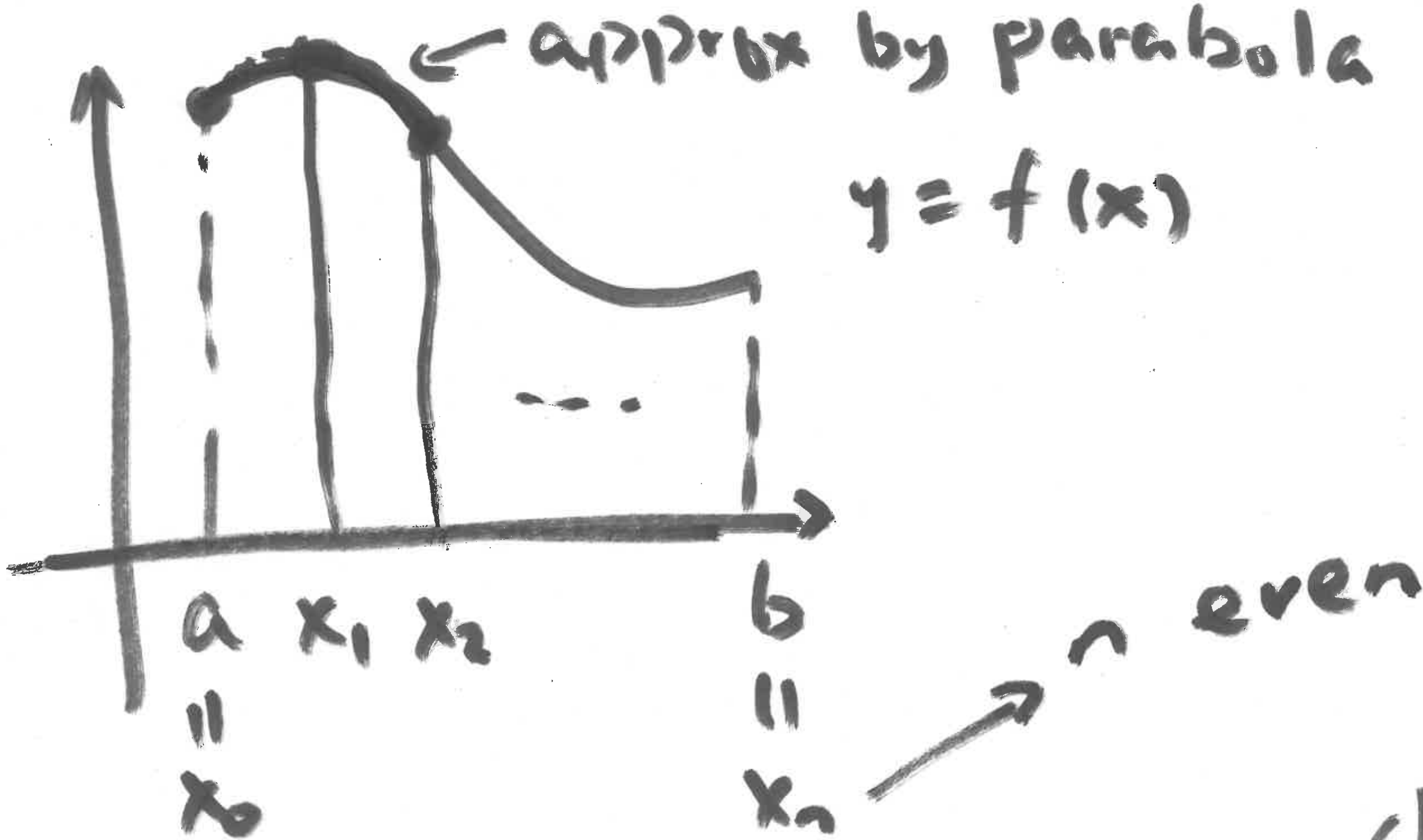
$$f(x)=\frac{1}{x} \qquad f'(x)=-\frac{1}{x^2}$$

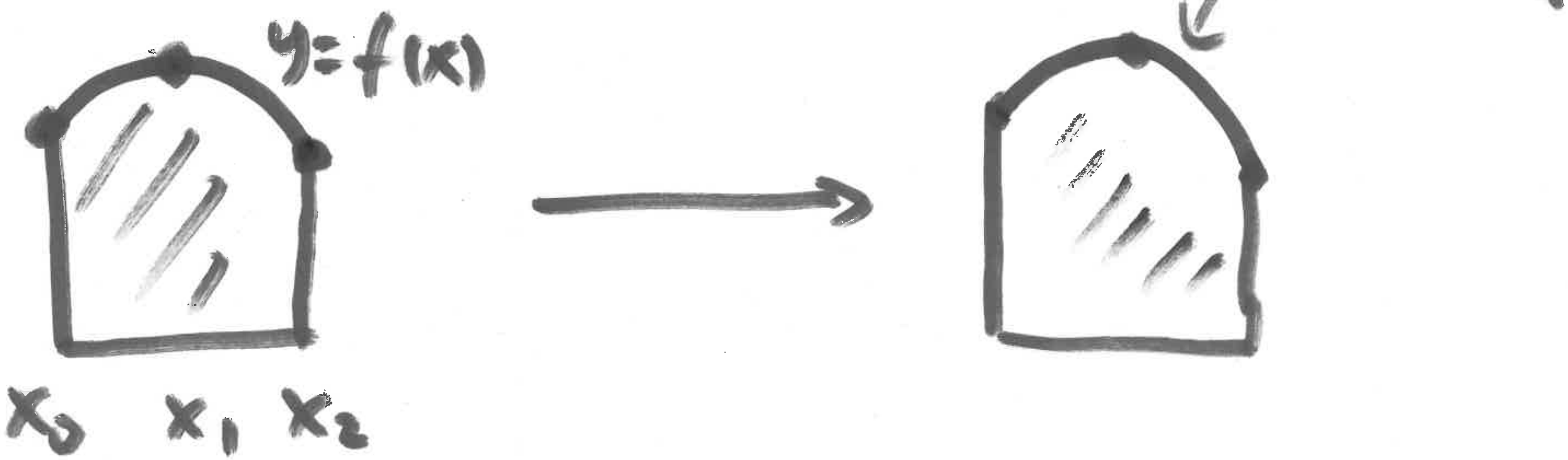
$$f''(x)=\frac{2}{x^3} \qquad \underline{1\le x\le 2}$$

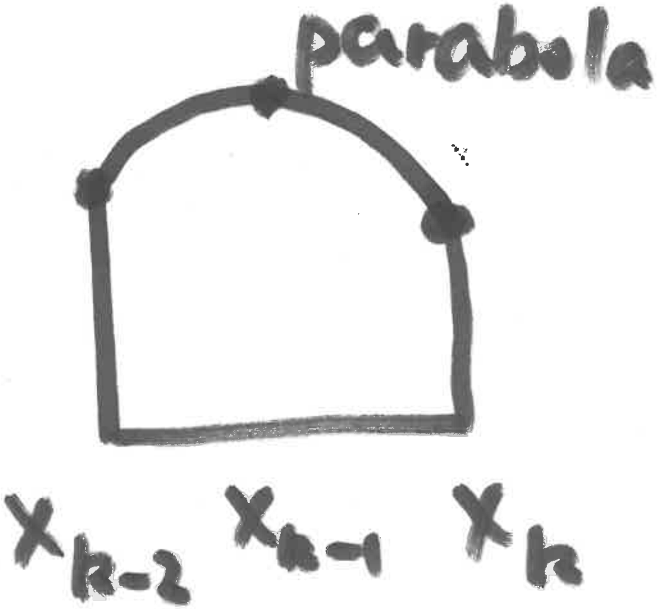
$$\max_{1\le x\le 2}|f''(x)|=2$$

$$E_4^T \le \frac{2}{12 \cdot 4^2} \cdot (2-1)^3 = \frac{1}{96}$$

$$E_4^T = 0.0038 \ldots$$


approx by parabola
y = f(x)
a
x1
x2
b
x0
xn
n even


y=f(x)
x0
x1
x2
parabola



$$\text{area} = \frac{b-a}{3n}\left[f(x_{k-2}) + 4f(x_{k-1}) + f(x_k)\right]$$

$$\int_a^b f(x)\,dx = \frac{b-a}{3n}\Big[\big(f(x_0) + 4f(x_1) + f(x_2)\big) + \big(f(x_2) + 4f(x_3) + f(x_4)\big) + \cdots + \big(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big)\Big]$$

$$= \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) \right.$$

$$\left. + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n) \right]$$

Simpson's rule

$$E_n^S \leq \frac{\max_{a \leq x \leq b} |f^{(4)}(x)|}{180 n^4} (b-a)^5$$

<u>ex 2</u> ------ Simpson's rule.

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{1}{3\cdot 4}\left[1 + 4\cdot\frac{4}{5} + 2\cdot\frac{2}{3} + 4\cdot\frac{4}{7} + \frac{1}{2}\right] = 0.693253\ldots$$

$$f'''(x) = -\frac{6}{x^4} \qquad f^{(4)}(x) = \frac{24}{x^5}$$

$$|f^{(4)}(x)| \le 24 \qquad 1 \le x \le 2$$

$$E_4^S \leq \frac{24}{180 \cdot 4^4}(2-1)^5 = 0.00013\cdots$$

$$E_4^S = 0.00010\cdots -$$