8.4 Partial fractions

rational function
(ratio of polynomials)

$\downarrow$

sum of polynomial,

$$\frac{A}{ax+b}, \quad \frac{Bx+C}{ax^2+bx+c}, \quad \ldots$$

① Make the fraction "proper"

(deg of num < deg of denom)

"long division"

<u>ex 1</u> $$\frac{x^4+2x^3-x}{x^2+3}$$

$$
\begin{array}{r l}
 & x^2 + 2x - 3 \\
\cline{2-2}
x^2+0x+3 \big) & x^4 + 2x^3 + 0 \cdot x^2 - x + 0 \\
 & x^4 + 0x^3 + 3x^2 \\
\cline{2-2}
 & \quad 2x^3 - 3x^2 - x \\
 & \quad 2x^3 + 0x^2 + 6x \\
\cline{2-2}
 & \qquad -3x^2 - 7x + 0 \\
 & \qquad -3x^2 + 0x - 9 \\
\cline{2-2}
 & \qquad\qquad -7x + 9
\end{array}
$$

$$\frac{x^4+2x^3-x}{x^2+3} = x^2+2x-3+\frac{-7x+9}{x^2+3}$$

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② Factor the denominator.

for each $(ax+b)^r$ factor,

write $\dfrac{A_1}{ax+b} + \cdots + \dfrac{A_r}{(ax+b)^r}$

for each $(ax^2+bx+c)^s$,

write $\dfrac{B_1x+C_1}{ax^2+bx+c} + \cdots + \dfrac{B_sx+C_s}{(ax^2+bx+c)^s}$

ex 2

$$\frac{x-2}{x^3+2x^2+x} = \frac{x-2}{x(x^2+2x+1)}$$

$$= \frac{x-2}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$x-2 = A\cdot(x+1)^2 + B\cdot x\cdot(x+1) + Cx$$

"Choose x to make linear factor $= 0$"

$x=0$: $-2 = A \cdot 1$ $\quad A = -2$

$x=-1$: $-1-2 = C \cdot (-1)$ $\quad C = 3$

x^2 coeff : $0 = A + B$ $\quad B = 2$

$$\frac{x-2}{x^3+2x^2+x} = \frac{-2}{x} + \frac{2}{x+1} + \frac{3}{(x+1)^2}$$

$$\int \frac{1}{1+x^3}\,dx$$

$$= \int \frac{1}{(1+x)(1-x+x^2)}\,dx$$

$$= \int \left(\frac{A}{1+x} + \frac{Bx+C}{1-x+x^2} \right) dx$$

$$= \frac{1}{3} \int \left(\frac{1}{1+x} + \frac{-x+2}{1-x+x^2} \right) dx$$

$$1 = A(1-x+x^2) + (Bx+C)(1+x)$$

$x = -1$: $1 = A(1-(-1)+1)$

$$A = \frac{1}{3}$$

x^2 coeff : $0 = A + B$ $\quad B = -\frac{1}{3}$

const coeff : $1 = A + C$

$$C = 1 - A = \frac{2}{3}$$

$$\int \frac{1}{1+x}\,dx = \ln|1+x| + C$$

$$\int \frac{-x+2}{1-x+x^2}\,dx = \int \frac{-x+2}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \int \frac{-\left(x-\frac{1}{2}\right)+\frac{3}{2}}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= -\int \frac{(x-\frac{1}{2})}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx$$

$$+ \frac{3}{2}\int \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx$$

two sub-problems

(1)

$$\int \frac{x-\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx$$

$$\left(\begin{array}{l} u = (x-\frac{1}{2})^2 \\ du = 2(x-\frac{1}{2})\,dx \end{array} \right)$$

$$= \frac{1}{2}\int \frac{1}{u+\frac{3}{4}}\,du$$

$$= \frac{1}{2}\ln\left|(x-\frac{1}{2})^2+\frac{3}{4}\right| + C$$

(2)

$$\int \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x-\frac{1}{2})\right)+C$$

" $\int \frac{1}{x^2+a^2}\,dx$ formula "

answer: $\frac{1}{3}\ln|1+x| - \frac{1}{6}\ln(x^2-x+1)$

$$+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$