# 8.3 Trig subs

Integrals with

$\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$, $\sqrt{x^2 - a^2}$

$\downarrow$ $\downarrow$ $\downarrow$

$x = a\sin u$ $x = a\tan u$ $x = a\sec u$

ex 1 $\int \sqrt{4-x^2}\,dx$

$$\left( \begin{array}{l} x = 2\sin u \\ dx = 2\cos u\,du \end{array} \right)$$

$$= \int \sqrt{4-4\sin^2 u} \cdot 2\cos u\,du$$

$$= 2\cdot 2 \int \sqrt{\cos^2 u}\,\cos u\,du$$

$$= 4 \int \cos^2 u\,du$$

$$= 4 \int \frac{1}{2}(1+\cos 2u)\, du$$

$$= 2u + \sin 2u + C$$

$$x = 2 \sin u$$

$$\sin u = \frac{x}{2}$$

$$u = \sin^{-1}\left(\frac{x}{2}\right)$$

$$\sin 2u = 2 \sin u \cos u$$

$$= 2 \cdot \frac{x}{2} \cdot \sqrt{1 - \left(\frac{x}{2}\right)^2}$$

answer $= 2\sin^{-1}\left(\frac{x}{2}\right)$

$+ x \cdot \sqrt{1 - \frac{x^2}{4}} + C$

ex 2

$$\int \frac{1}{\sqrt{x^2+2x+10}}\,dx$$

$$= \int \frac{1}{\sqrt{(x+1)^2+9}}\,dx$$

$$\left( \begin{array}{l} x+1 = 3\tan u \\ dx = 3 \cdot \sec^2 u \, du \end{array} \right)$$

$$= \int \frac{1}{\sqrt{9\tan^2 u + 9}} \cdot 3\sec^2 u \, du.$$

$$= \int \frac{1}{\sec u} \sec^2 u \, du$$

$$= \int \sec u \, du$$

$$= \ln|\sec u + \tan u| + C$$

$$\tan u = \frac{x+1}{3}$$

$$\sec u = \sqrt{\left(\frac{x+1}{3}\right)^2 + 1}$$

$$\text{answer} = \ln\left|\sqrt{\left(\frac{x+1}{3}\right)^2+1} + \frac{x+1}{3}\right| + C$$

ex 3

$$\int \frac{\sqrt{x^2-9}}{x}\,dx$$

$$\left( \begin{aligned} x &= 3\sec u \\ dx &= 3\sec u \tan u\, du \end{aligned} \right)$$

$$= \int \frac{\sqrt{9\sec^2 u - 9}}{3\sec u}\, 3\sec u \tan u\, du$$

$$= 3\int \tan^2 u\, du$$

$$= 3\int (\sec^2 u - 1)\,du$$

$$= 3(\tan u - u) + C$$

$$x = 3\sec u$$

$$\sec u = \frac{x}{3}$$

$$\cos u = \frac{3}{x}$$

$$u = \cos^{-1}\left(\frac{3}{x}\right)$$

$$\tan u = \sqrt{\sec^2 u - 1}$$

$$= \sqrt{\left(\frac{x}{3}\right)^2 - 1}$$

answer $= 3\left(\sqrt{\left(\frac{x}{3}\right)^2 - 1} - \cos^{-1}\left(\frac{3}{x}\right)\right) + C$

method 2:

$$\int \frac{\sqrt{x^2-9}}{x}\,dx$$

$$= \int \frac{\sqrt{x^2-9}}{x^2}\, x\,dx$$

$$\left( \begin{aligned} u &= x^2-9 \\ du &= 2x\,dx \end{aligned} \right)$$

$$= \frac{1}{2}\int \frac{\sqrt{u}}{u+9}\,du$$

$$\left(\begin{array}{c} v = \sqrt{u} \\ v^2 = u \\ 2v\,dv = du \end{array}\right)$$

$$= \frac{1}{2}\int \frac{v}{v^2+9}\,2v\,dv$$

$$= \int \frac{v^2}{v^2+9}\,dv$$

$$= \int \frac{v^2+9 \; -9}{v^2+9} \, dv$$

$$= \int \left( 1 - 9 \cdot \frac{1}{v^2+9} \right) dv$$

$$= v - 9 \cdot \frac{1}{3} \tan^{-1}\left(\frac{v}{3}\right) + C$$

$$= \sqrt{u} - 3 \tan^{-1}\left(\frac{\sqrt{u}}{3}\right) + C$$

$$= \sqrt{x^2-9} - 3 \tan^{-1}\left(\frac{\sqrt{x^2-9}}{3}\right) + C$$

**ex 4** $\int \sqrt{x(1-x^3)}\, dx$

$$= \int \sqrt{x} \cdot \sqrt{1-x^3}\, dx$$

$$\left( \begin{array}{l} u = x^{3/2} \\ du = \frac{3}{2}\sqrt{x}\, dx \end{array} \right)$$

$$= \frac{2}{3} \int \sqrt{1-u^2}\, du$$

$$\left(\begin{array}{l} u = \sin v \\ du = \cos v \, dv \end{array}\right)$$

$$= \frac{2}{3}\int \cos^2 v \, dv$$

$$= \frac{2}{3}\int \frac{1}{2}(1+\cos 2v)\, dv$$

$$= \frac{1}{3}v + \frac{1}{3}\cdot\frac{1}{2}\sin 2v + C$$

$$= \frac{1}{3}\sin^{-1}(x^{3/2}) + \frac{1}{3}x^{3/2}\sqrt{1-x^3} + C$$