8.2 Trig integrals

$\int \sin^m x \cos^n x \, dx$

either m or n is

odd and positive

m is odd positive $\Rightarrow u = \cos x$

n .... $\Rightarrow u = \sin x$

ex 1 $\int \sin^3 x \cos^2 x \, dx$

$$\begin{pmatrix} u = \cos x \\ du = -\sin x \, dx \end{pmatrix}$$

$$= \int \sin^2 x \cos^2 x \cdot \sin x \, dx$$

$$= \int (1 - \cos^2 x) \cos^2 x \cdot \sin x \, dx$$

$$= -\int (1 - u^2) u^2 \, du$$

$$= -\int (u^2 - u^4) \, du$$

$$= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C.$$

$$\int \cot x \cdot \csc^4 x \, dx$$

$$= \int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin^4 x} \, dx$$

$$= \int \frac{\cos x}{\sin^5 x} \, dx$$

$$\left(\begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \right)$$

$$= \int \frac{1}{u^5} \, du$$

$$= \frac{1}{-4} u^{-4} + C$$

$$= -\frac{1}{4} \cdot \frac{1}{\sin^4 x} + C$$

$$\int \sin^m x \cos^n x \, dx$$

m, n even, positive

$$\Rightarrow \begin{cases} \cos^2 x = \frac{1}{2}(1 + \cos 2x) \\ \sin^2 x = \frac{1}{2}(1 - \cos 2x) \end{cases}$$

ex 2 $\int \sin^2 x \cos^2 x \, dx$

$$= \int \frac{1}{2}(1 - \cos 2x) \cdot \frac{1}{2}(1 + \cos 2x)\, dx$$

$$= \frac{1}{4} \int (1 - \cos^2 2x)\, dx$$

$$= \frac{1}{4} \int \left(1 - \frac{1}{2}(1 + \cos 4x)\right) dx$$

$$= \frac{1}{4} \int \left(\frac{1}{2} - \frac{1}{2} \cos 4x\right) dx$$

$$= \frac{1}{8}x - \frac{1}{8}\cdot\frac{1}{4}\sin 4x + C$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

$$\int \tan^m x \cdot \sec^n x \, dx$$

m, n even, positive

$\Rightarrow$ $u = \tan x$

$du = \sec^2 x \, dx$

__ex 3__

$$\int \tan^2 x \sec^4 x \, dx$$

$$\left(\begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \end{array} \right)$$

$$= \int \tan^2 x \sec^2 x \cdot \sec^2 x \, dx$$

$$= \int \tan^2 x \left(\tan^2 x + 1 \right) \cdot \sec^2 x \, dx$$

$$= \int u^2 (u^2+1)\, du$$

$$= \int (u^4 + u^2)\, du$$

$$= \frac{1}{5} u^5 + \frac{1}{3} u^3 + C$$

$$= \frac{1}{5} \tan^5 x + \frac{1}{3} \tan^3 x + C$$

$\int \tan^m x \sec^n x \, dx$

m even, positive

n odd, positive

$\Rightarrow$ first write $\tan^2 x = \sec^2 x - 1$

then $\boxed{\begin{array}{l} \int \sec^n x \, dx \\ n \text{ odd} \end{array}}$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x \, dx = \int \sec x \cdot \sec^2 x \, dx$$

$$\left(\begin{array}{ll} u = \sec x & v = \tan x \\ du = \sec x \cdot \tan x \, dx & dv = \sec^2 x \, dx \end{array} \right)$$

$$= \sec x \cdot \tan x - \int \tan x \cdot \sec x \cdot \tan x \, dx$$

$$= \sec x \cdot \tan x - \int \sec x \cdot \tan^2 x \, dx$$

$$= \sec x \cdot \tan x - \int \sec x \cdot (\sec^2 x - 1)\, dx$$

$$= \sec x \cdot \tan x - \int \sec^3 x\, dx$$

$$+ \int \sec x\, dx$$

$$\Rightarrow 2 \int \sec^3 x\, dx = \sec x \tan x$$

$$+ \ln | \sec x + \tan x |$$

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x$$

$$+ \frac{1}{2} \ln | \sec x + \tan x | + C .$$

ex 4

$$\int \frac{\sin^4 x}{\cos^2 x}\,dx$$

$$= \int \frac{(1-\cos^2 x)^2}{\cos^2 x}\,dx$$

$$= \int \frac{1-2\cos^2 x+\cos^4 x}{\cos^2 x}\,dx$$

$$= \int (\sec^2 x - 2 + \cos^2 x)\,dx$$

$$= \tan x - 2x + \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

$$\int \sin ax \cos bx \, dx$$

$$\sin ax \cos bx = \frac{1}{2}\left(\sin(a+b)x\right.$$

$$\left. + \sin(a-b)x\right)$$

ex 5 $\int \sin 3x \cos 7x \, dx$

$$= \int \frac{1}{2}\left(\sin 10x + \sin(-4x)\right) dx$$

$$= \frac{1}{2}\left(-\frac{1}{10}\cos 10x + \frac{1}{4}\cos 4x\right) + C$$

$$= -\frac{1}{20}\cos 10x + \frac{1}{8}\cos 4x + C.$$