SOLUTION

Problem 1. (25 points) Let R be the region in the first quadrant bounded by $y = x^2$, $x = 0$, $y = 4$. Compute the volume by revolving R around the line $x = -1$.

We first write $y = x^2$ into $x = \sqrt{y}$. Then

$$V = \int_0^4 \pi((\sqrt{y}+1)^2 - 1^2)\,\mathrm{d}y = \pi\int_0^4 (y + 2\sqrt{y})\,\mathrm{d}y = \pi(\frac{1}{2}y^2 + \frac{4}{3}y^{3/2})|_0^4 = \pi(8 + \frac{32}{3}) = \frac{56\pi}{3}$$

Problem 2. (25 points) There is a water tank whose shape is a cone, whose vertex is at bottom. Its base radius is 10 feet, and height is 5 feet. At the beginning the tank is full of water. If you want to pump out all the water to the level which is 2 feet higher than the top of the tank, how much work (in foot-pounds) do you need to do? (The density of water is 62.5 pounds/foot^3)

From picture, the cross-section is a circle, with radius r satisfying

$$\frac{r}{x} = \frac{10}{5} \quad \Rightarrow \quad r = 2x$$

Therefore

$$W = \int_0^5 62.5\cdot(7-x)\cdot\pi(2x)^2\,\mathrm{d}x = 250\pi\int_0^5 (7x^2 - x^3)\,\mathrm{d}x = 250\pi(\frac{7}{3}x^3 - \frac{1}{4}x^4)|_0^5 = 250\pi(\frac{7}{3}5^3 - \frac{1}{4}5^4)$$

Problem 3. (25 points) Let P be a thin plate, whose mass is uniformly distributed, and whose shape is bounded by the x-axis and the graph of $y = 1 - |x|$. Compute its center of gravity.

$$M_x = \frac{1}{2}\int_{-1}^1 (1 - |x|)^2\,\mathrm{d}x$$

Since $1 - |x| = 1 - x$ for $x \geq 0$ and $1 - |x| = 1 + x$ for $x < 0$, one can separate the integral into two parts:

$$M_x = \frac{1}{2}(\int_{-1}^0 (1+x)^2\,\mathrm{d}x + \int_0^1 (1-x)^2\,\mathrm{d}x) = \frac{1}{2}(\frac{1}{3}(1+x)^3|_{-1}^0 + \frac{1}{3}(x-1)^3|_0^1) = \frac{1}{3}$$

The area $A = \frac{1}{2}2\cdot 1 = 1$ since it is a triangle. Therefore

$$\bar{y} = \frac{1}{3}$$

By symmetry (or by computing $M_y = \int_{-1}^{1} x(1-|x|)\,\mathrm{d}x = 0$)

$$\bar{x} = 0$$

Problem 4. (25 points) (1) (15 points) Let C be the parametric curve determined by

$$\begin{cases} x = 3 + \sin^2 t \\ y = 2 + 2\cos^2 t \end{cases} \qquad 0 \le t \le 2\pi$$

Derive the equation of C (as an equation in x and y) and describe the shape of C.
(2) (10 points) What is the length of the part of C with $0 \le t \le \frac{\pi}{6}$?

(1) Multiply the first equation by 2 and add with the second equation and get

$$2x + y = 6 + 2\sin^2 t + 2 + 2\cos^2 t = 10$$

or

$$y = 10 - 2x$$

which is the equation of a line. Notice that $0 \le \sin^2 t \le 1$, therefore x ranges from 3 to 4. Therefore C is the part of the line $y = 10 - 2x$ with $3 \le x \le 4$ (or, the line segment from (3,4) to (4,2)).
(2)

$$f(t) = 3 + \sin^2 t, \quad f'(t) = 2\sin t\cos t$$
$$g(t) = 2 + 2\cos^2 t, \quad g'(t) = -4\sin t\cos t$$

$$L = \int_0^{\frac{\pi}{6}} \sqrt{4\sin^2 t\cos^2 t + 16\sin^2 t\cos^2 t}\,\mathrm{d}t = \int_0^{\frac{\pi}{6}} \sqrt{20}\sin t\cos t\,\mathrm{d}t = \frac{\sqrt{20}}{2}\sin^2 t\Big|_0^{\frac{\pi}{6}} = \frac{\sqrt{20}}{8} = \frac{\sqrt{5}}{4}$$

or, this part of C is the line segment from (3,2+2) to $(3+\frac{1}{4}, 2+\frac{3}{2})$, therefore

$$L = \sqrt{(\frac{1}{4})^2 + (\frac{1}{2})^2} = \frac{\sqrt{5}}{4}$$