ex

$$\int \frac{1}{1+x+x^2}\,dx$$

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \int \frac{1}{\frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right)}\,dx$$

$$\left(\begin{aligned} u &= \sqrt{\tfrac{4}{3}}\left(x+\tfrac{1}{2}\right) \\ du &= \sqrt{\tfrac{4}{3}}\, dx \end{aligned} \right)$$

$$= \frac{4}{3}\sqrt{\frac{3}{4}} \int \frac{1}{u^2+1}\, du$$

$$= \sqrt{\frac{4}{3}} \tan^{-1} u + C$$

$$= \sqrt{\frac{4}{3}} \tan^{-1}\left(\sqrt{\frac{4}{3}}\left(x+\frac{1}{2}\right)\right) + C.$$

$$\int \frac{1}{\sqrt{4x - x^2}} \, dx$$

$$= \int \frac{1}{\sqrt{-x^2 + 4x - 4 + 4}} \, dx$$

$$= \int \frac{1}{\sqrt{4 - (x-2)^2}} \, dx$$

$$= \int \frac{1}{\sqrt{4\left(1 - \frac{1}{4}(x-2)^2\right)}} \, dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{1 - \frac{1}{4}(x-2)^2}} \, dx$$

$$\left(\begin{array}{l} u = \frac{1}{2}(x-2) \\ \\ du = \frac{1}{2} \, dx \end{array} \right)$$

$$= \int \frac{1}{\sqrt{1-u^2}} \, du = \sin^{-1} u + C$$

$$= \sin^{-1}\left(\frac{1}{2}(x-2) \right) + C .$$

7.6 L'Hôpital's rule

$$\lim_{x \to *} \frac{f(x)}{g(x)} = \lim_{x \to *} \frac{f'(x)}{g'(x)}$$

if "$\frac{0}{0}$" / "$\frac{\infty}{\infty}$", and

RHS exists (number / ∞)

Proof for * is a number a,

"$\frac{0}{0}$", $g'(a) \neq 0$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

$$= \lim_{x \to a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} = \frac{f'(a)}{g'(a)}$$

$$= \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

ex 1

$$\lim_{x\to 0} \frac{\sin 2x}{e^x - 1} \underset{\uparrow}{=} \lim_{x\to 0} \frac{2\cos 2x}{e^x}$$

"$\frac{0}{0}$"

$$= \frac{2}{1} = 2.$$

$$\lim_{x\to \frac{\pi}{2}} \frac{\cos x}{\cos 3x} \underset{\uparrow}{=} \lim_{x\to \frac{\pi}{2}} \frac{-\sin x}{-3\sin 3x} = \frac{-1}{3}$$

"$\frac{0}{0}$"

$$\lim_{x\to\infty} \frac{x^2}{e^x} \underset{\substack{\uparrow \\ \text{"}\frac{\infty}{\infty}\text{"}}}{=} \lim_{x\to\infty} \frac{2x}{e^x}$$

$$\underset{\substack{\uparrow \\ \text{"}\frac{\infty}{\infty}\text{"}}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

- "$0 \cdot \infty$" — flip one of them to denominator

$$\lim_{x \to 0+} x \ln x = \lim_{x \to 0+} \frac{\ln x}{1/x}$$

$$\underset{\substack{\uparrow \\ \text{"}\frac{\infty}{\infty}\text{"}}}{=} \lim_{x \to 0+} \frac{1/x}{-1/x^2} = \lim_{x \to 0+} \frac{x}{-1}$$

$$= 0$$

$$\lim_{x \to \infty} x\left(\frac{\pi}{2} - \tan^{-1} x\right)$$

$$= \lim_{x \to \infty} \frac{\frac{\pi}{2} - \tan^{-1} x}{1/x}$$

$$\underset{\uparrow}{=} \lim_{x \to \infty} \frac{-1/(1+x^2)}{-1/x^2}$$

"$\frac{0}{0}$"

$$= \lim_{x \to \infty} \frac{x^2}{1+x^2} = 1$$