7.2 Natural exponential

$f(x) = e^x$

$e \approx 2.71828\ldots$

$y = e^x$

$\Updownarrow$

$x = \ln y$

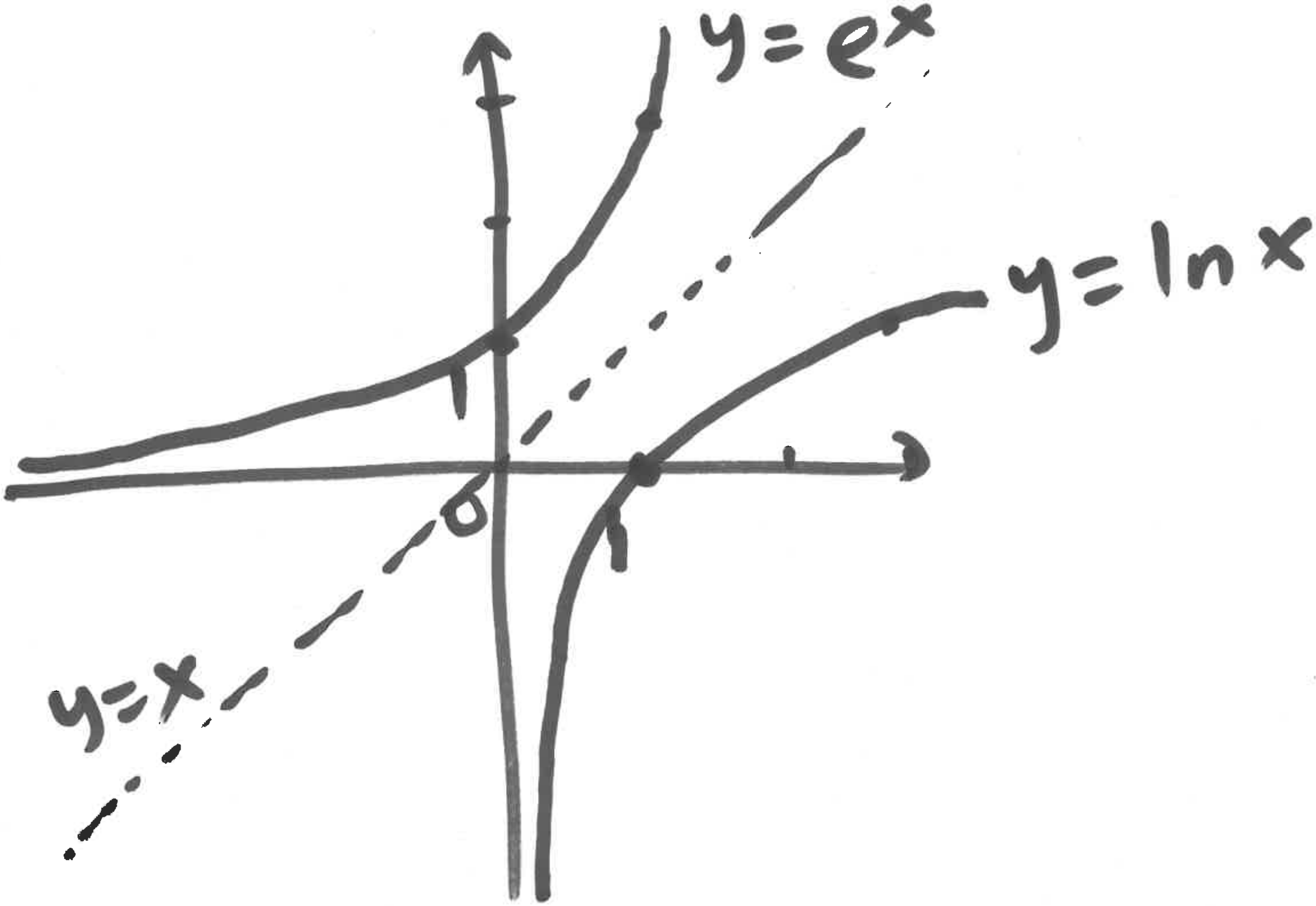
$f(0) = 1$

$f(1) = e$

$f(-1) = \frac{1}{e}$

The inverse of $f(x) = e^x$

is $f^{-1}(x) = \ln x$



domain of e^x: $(-\infty, \infty)$

range of e^x: $(0, \infty)$

domain of $\ln x$: $(0, \infty)$

range of $\ln x$: $(-\infty, \infty)$

- $e^{\ln x} = x \quad (x > 0)$

  $\ln(e^{x}) = x$

- $e^{a+b} = e^{a} \cdot e^{b}$

  $\ln(ab) = \ln a + \ln b \quad (a, b > 0)$

- $e^{-a} = \dfrac{1}{e^{a}}$

  $\ln\left(\frac{1}{a}\right) = -\ln a$

  $\ln(a^{b}) = b \ln a$

- $(e^x)' = e^x$, $(\ln x)' = \dfrac{1}{x}$

derive:

$f(x) = e^x$ $\qquad f^{-1}(x) = \ln x$

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(\ln b)}$$

$b = f(a)$

$$= \frac{1}{e^{\ln b}}$$

$b = e^a$ $\qquad a = \ln b$

$$= \boxed{\frac{1}{b}}$$

- $\int e^x \, dx = e^x + C$

- $$\lim_{x \to -\infty} e^x = 0$$

$$\lim_{x \to \infty} e^x = \infty$$

$$\lim_{x \to 0+} \ln x = -\infty$$

$$\lim_{x \to \infty} \ln x = \infty$$

ex 1 Sketch the graph of

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

"hyperbolic tangent" "tanh"

domain : $(-\infty, \infty)$

$$f'(x) = \frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$

$$= \frac{e^{2x}+2+e^{-2x}-(e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2}$$

$$= \frac{4}{(e^x+e^{-x})^2} > 0$$

$\Rightarrow$ f increasing

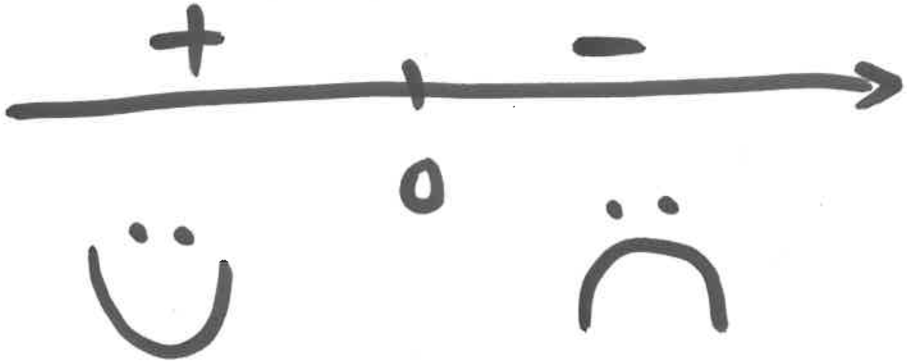
$$f''(x) = 4 \cdot (-2) \frac{1}{(e^x + e^{-x})^3} \cdot (e^x - e^{-x})$$

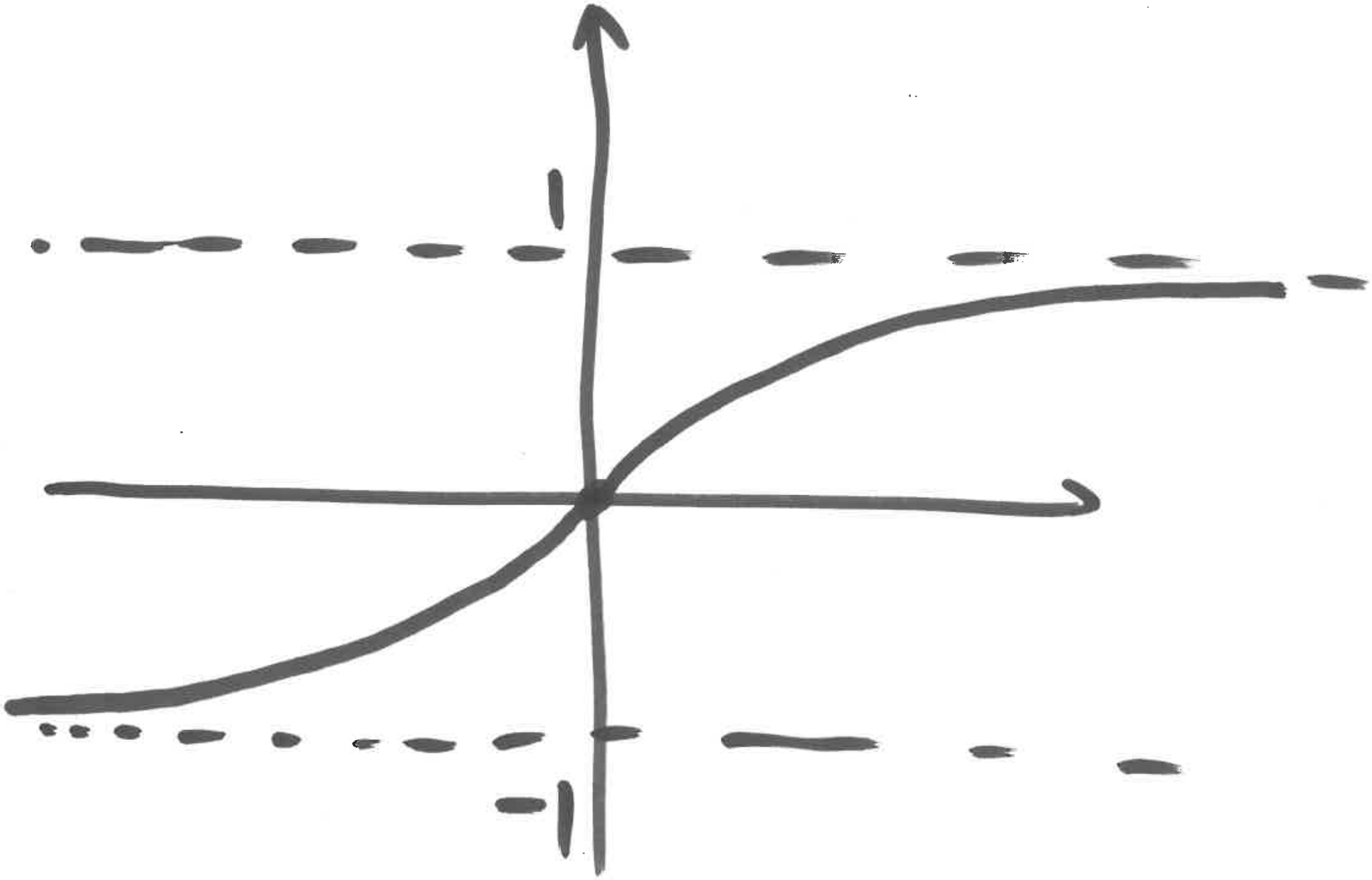
$$= -8 \cdot \frac{e^x - e^{-x}}{(e^x + e^{-x})^3}$$

$$f''(x) = 0 \Rightarrow e^x - e^{-x} = 0$$

$$e^x = e^{-x} \qquad x = -x \qquad \underline{x = 0}$$

f'' :





$$\lim_{x \to -\infty} f(x) = -1 \qquad f(0) = 0$$

$$\lim_{x \to \infty} f(x) = 1$$

ex 2

$$\frac{1}{2}\int 2\cdot\frac{e^{4\sqrt{x}}}{\sqrt{x}}\,dx$$

$$u = 4\sqrt{x}$$

$$du = 4\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x}} = 2\cdot\frac{1}{\sqrt{x}}\,dx$$

$$= \frac{1}{2}\int e^{u}\,du = \frac{1}{2}e^{u} + C$$

$$= \frac{1}{2}e^{4\sqrt{x}} + C$$

<u>ex 3</u> $\frac{d^n}{dx^n}\left(e^{-x^2}\right) = e^{-x^2} P_n(x)$

$P_n(x)$: polynomial

Derive formula for $P_{n+1}(x)$

from $P_n(x)$

$$e^{-x^2} P_{n+1}(x) = \frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right)$$

$$= \frac{d}{dx}\left(\frac{d^n}{dx^n}\left(e^{-x^2}\right)\right)$$

$$= \frac{d}{dx}\left(e^{-x^2} P_n(x)\right)$$

$$= e^{-x^2} \cdot (-2x) \cdot P_n(x) + e^{-x^2} P_n'(x)$$

$$= e^{-x^2}\left(-2x\, P_n(x) + P_n'(x)\right)$$

$$P_{n+1}(x) = -2x\, P_n(x) + P_n'(x)$$

$$P_0(x) = 1 \qquad P_1(x) = -2x$$

$$P_2(x) = -2x \cdot (-2x) + (-2) = 4x^2 - 2$$