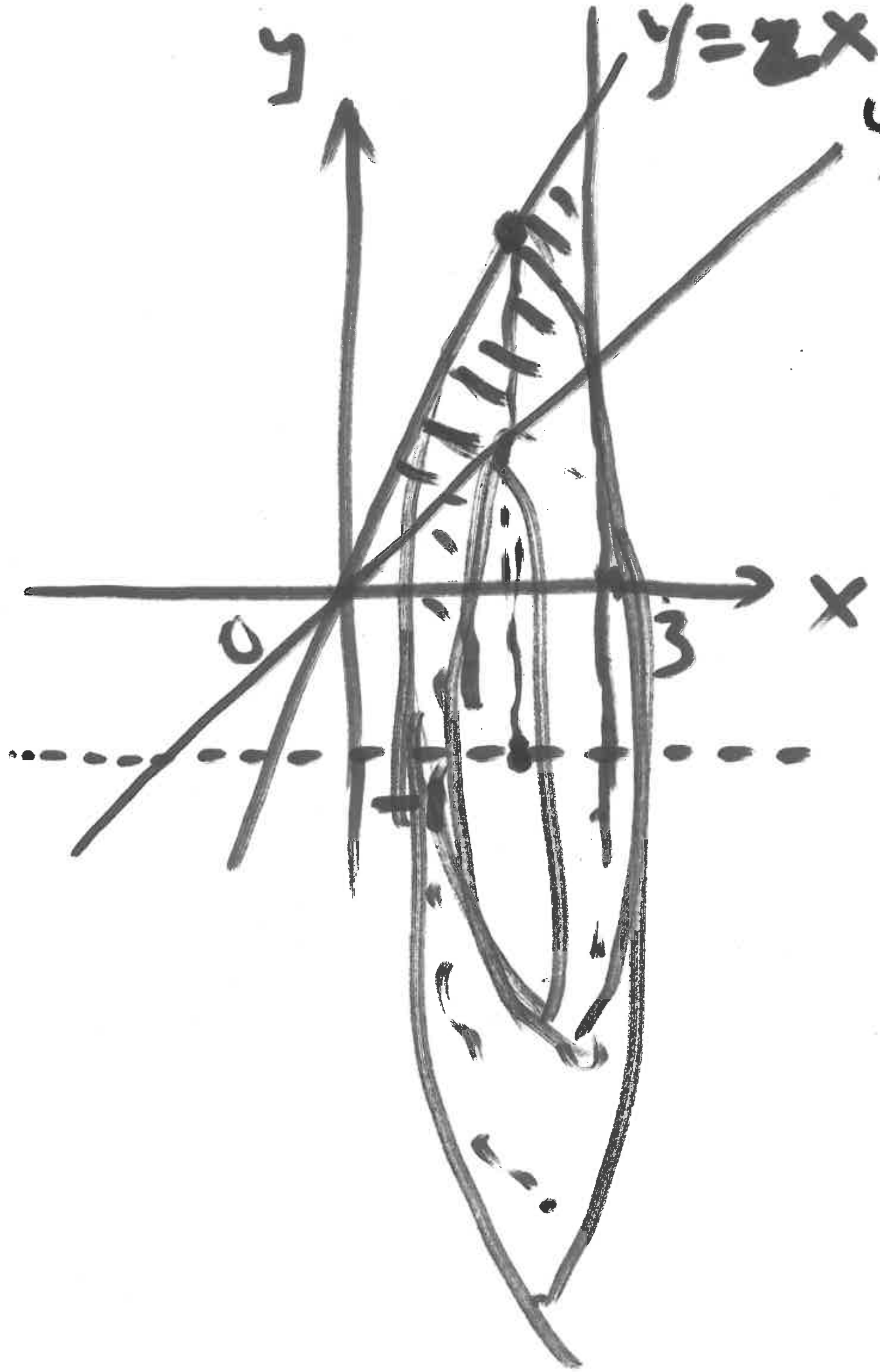
Midterm 1 Friday

- arrive 5 minutes earlier
- no calculator / cell phone / formula sheet

1. R is the region bounded by $y=x$, $y=2x$, $x=3$.

Compute the volume by revolving R around $y=-1$



$$A(x) = \pi\cdot(2x+1)^2 - \pi\cdot(x+1)^2$$

$$V = \int_0^3 \pi \left((2x+1)^2 - (x+1)^2 \right) dx$$

$$= \pi \int_0^3 \left(4x^2 + 4x + 1 - x^2 - 2x - 1 \right) dx$$

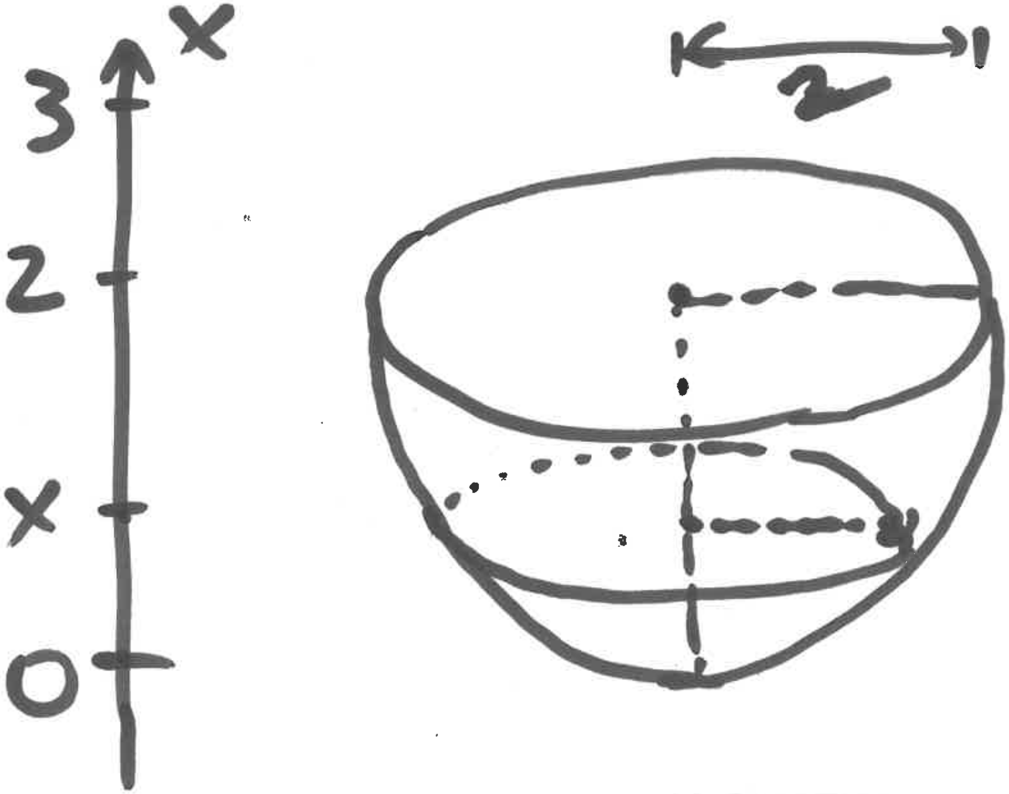
$$= \pi \int_0^3 (3x^2 + 2x)\, dx$$

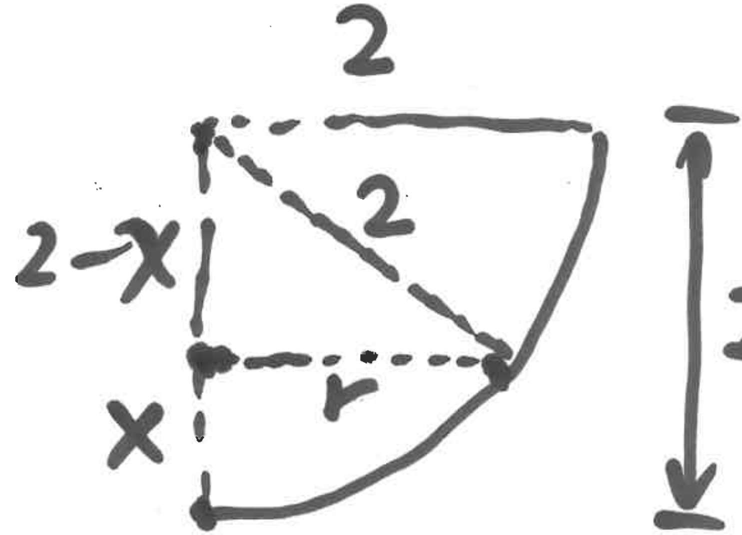
$$= \pi (x^3 + x^2) \Big|_0^3$$

$$= \pi (3^3 + 3^2) = \boxed{36\pi}$$

2. A hemi-sphere shaped water tank, opening-up, radius 2 feet, filled with water. What is the work by pumping out all water to 1 feet higher than its top?

water density 62.5 pounds/foot^3





$$r = \sqrt{2^2 - (2-x)^2}$$

$$= \sqrt{4 - (4 - 4x + x^2)}$$

$$r = \sqrt{4x - x^2}$$

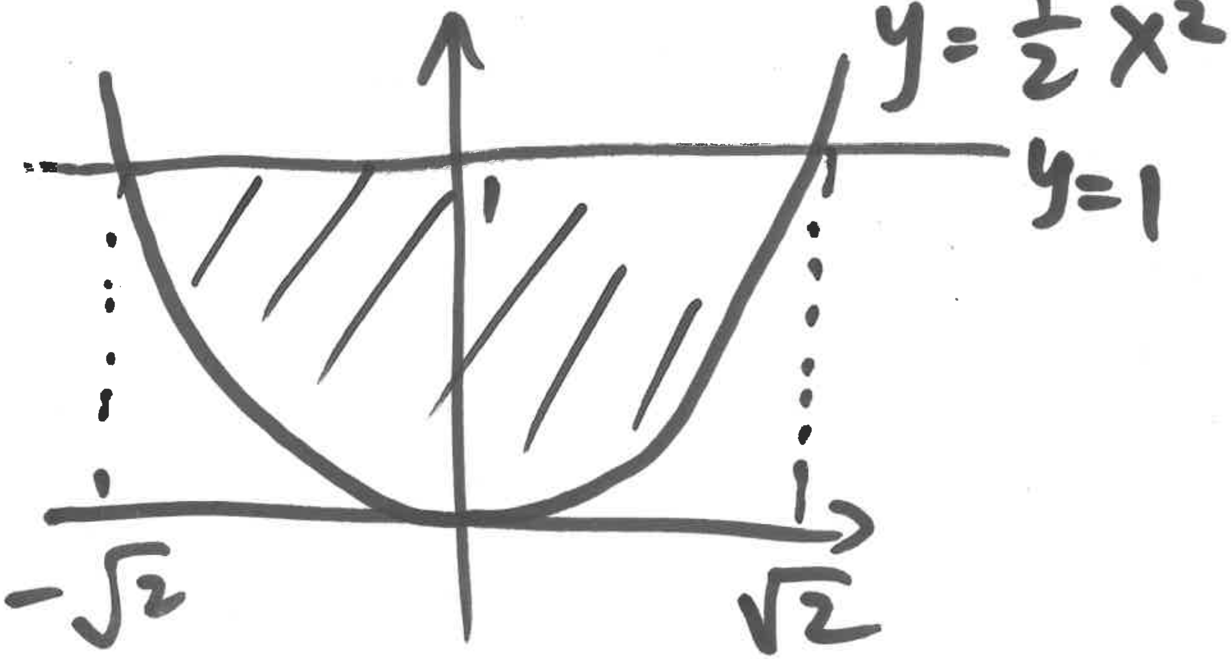
$$W = \int_0^2 62.5 \underbrace{(3-x)}_{\text{"}\ell - x\text{"}} \cdot \underbrace{\pi(4x-x^2)}_{\text{"}A(x)\text{"}} \, dx$$

$$= 62.5\pi \int_0^2 (12x - 4x^2 - 3x^2 + x^3)\,dx$$

$$= 62.5\pi \left(6x^2 - \frac{7}{3}x^3 + \frac{1}{4}x^4\right)\Big|_0^2$$

$$= \underline{62.5\pi \left(6\cdot 2^2 - \frac{7}{3}\cdot 2^3 + \frac{1}{4}2^4\right)}$$

3. A thin plate (mass uniformly distributed) bounded by $y = \frac{1}{2}x^2$, $y = 1$. Compute center of gravity.



$$\begin{cases} y = \frac{1}{2}x^2 \\ y = 1 \end{cases}$$

$$\frac{1}{2}x^2 = 1$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

$$M_x = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{2}\left(1^2 - \left(\frac{1}{2}x^2\right)^2\right) dx$$

$$= \frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} \left(1 - \frac{1}{4}x^4\right) dx$$

$$= \frac{1}{2}\left(x - \frac{1}{4}\cdot\frac{1}{5}x^5\right)\Big|_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \frac{1}{2}\left[\left(\sqrt{2} - \frac{1}{4}\cdot\frac{1}{5}4\sqrt{2}\right) - \left(-\sqrt{2} - \frac{1}{4}\cdot\frac{1}{5}\cdot(-4\sqrt{2})\right)\right]$$

$$= \frac{1}{2}\left(\frac{4}{5}\sqrt{2} + \frac{4}{5}\sqrt{2}\right)$$

$$= \frac{4}{5}\sqrt{2}$$

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} \left(1 - \frac{1}{2}x^2\right) dx$$

$$= \cdots = \frac{4}{3}\sqrt{2}$$

$$\bar{y} = \frac{M_x}{A} = \frac{\frac{4}{5}\sqrt{2}}{\frac{4}{3}\sqrt{2}} = \frac{3}{5}$$

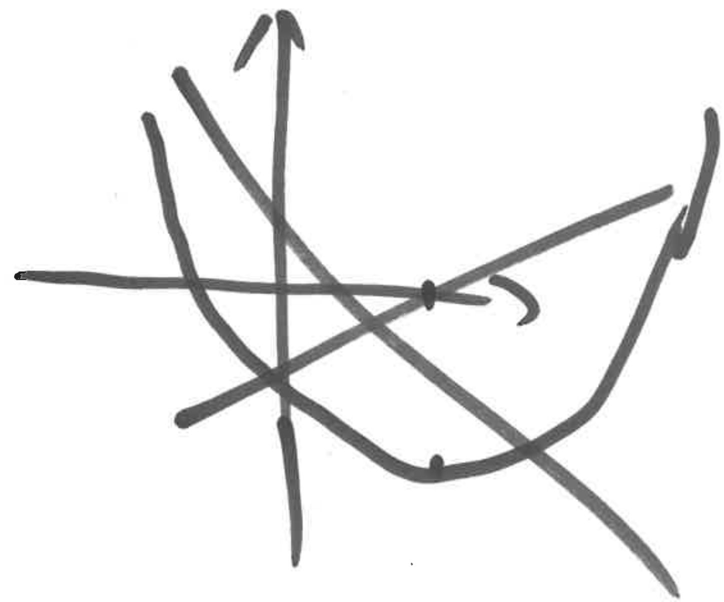
$$\bar{x} = 0 \quad (\text{Symmetry})$$

4. $\begin{cases} x = t^2+1 \\ y = t^4-2 \end{cases}$ shape? $\rightsquigarrow$ "$x \geq 1$"

$$t^2 = x-1$$

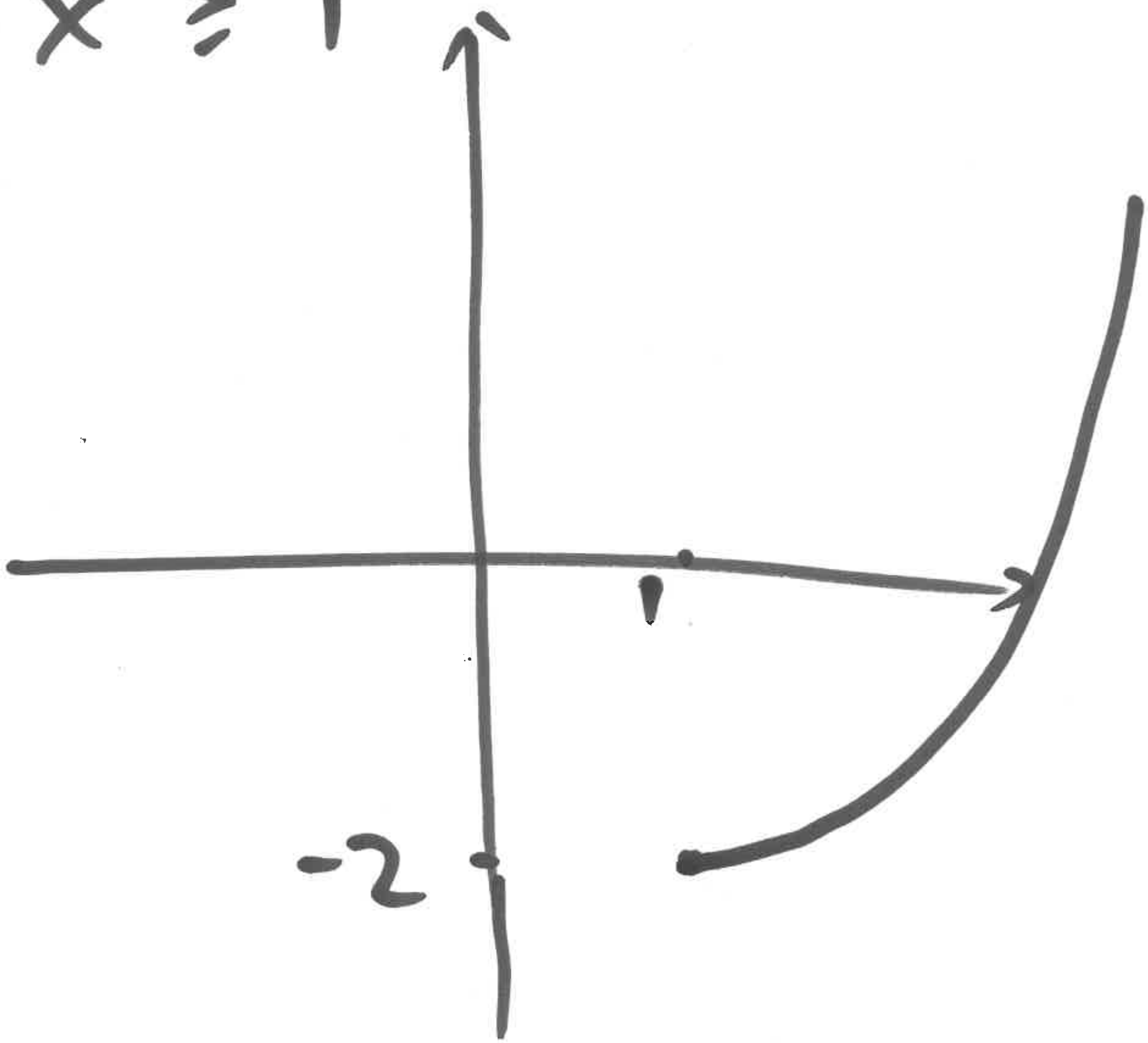
$$y = (t^2)^2 - 2 = (x-1)^2 - 2.$$

~~parabola~~.



part of parabola with

$x \geq 1$



(I planned this, but out of time...)

What is the curve length of the part with $0 \le t \le 1$?

(assume you know $a = \int_0^2 \sqrt{1+x^2}\,dx$)

$f(t) = t^2+1 \qquad f'(t) = 2t$

$g(t) = t^4-2 \qquad g'(t) = 4t^3$

$$L = \int_0^1 \sqrt{(2t)^2 + (4t^3)^2}\, dt$$

$$= 2\int_0^1 \sqrt{t^2 + 4t^6}\, dt$$

$$= 2\int_0^1 t\sqrt{1 + 4t^4}\, dt$$

$u = 2t^2$

$du = 4t\,dt$

$t: 0 \rightsquigarrow 1$

$u: 0 \rightsquigarrow 2$

$$= \frac{1}{2}\int_0^2 \sqrt{1+u^2}\, du$$

$$= \frac{1}{2} a.$$