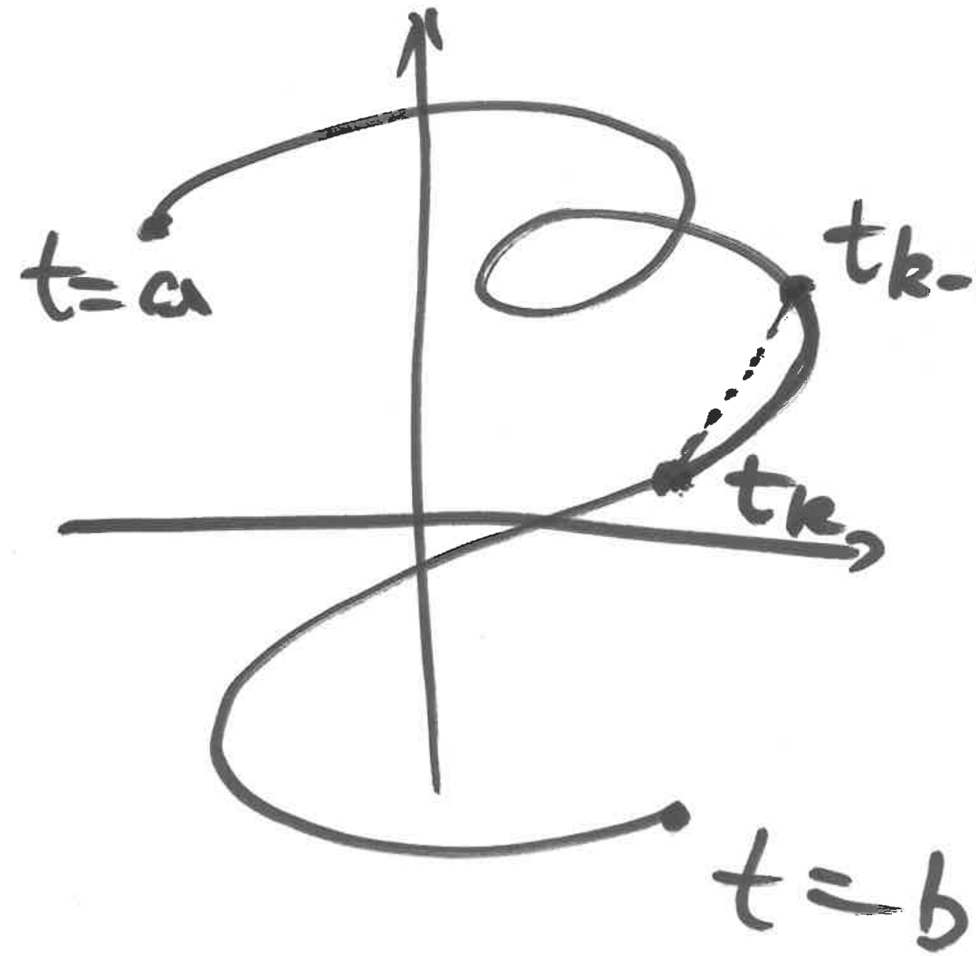
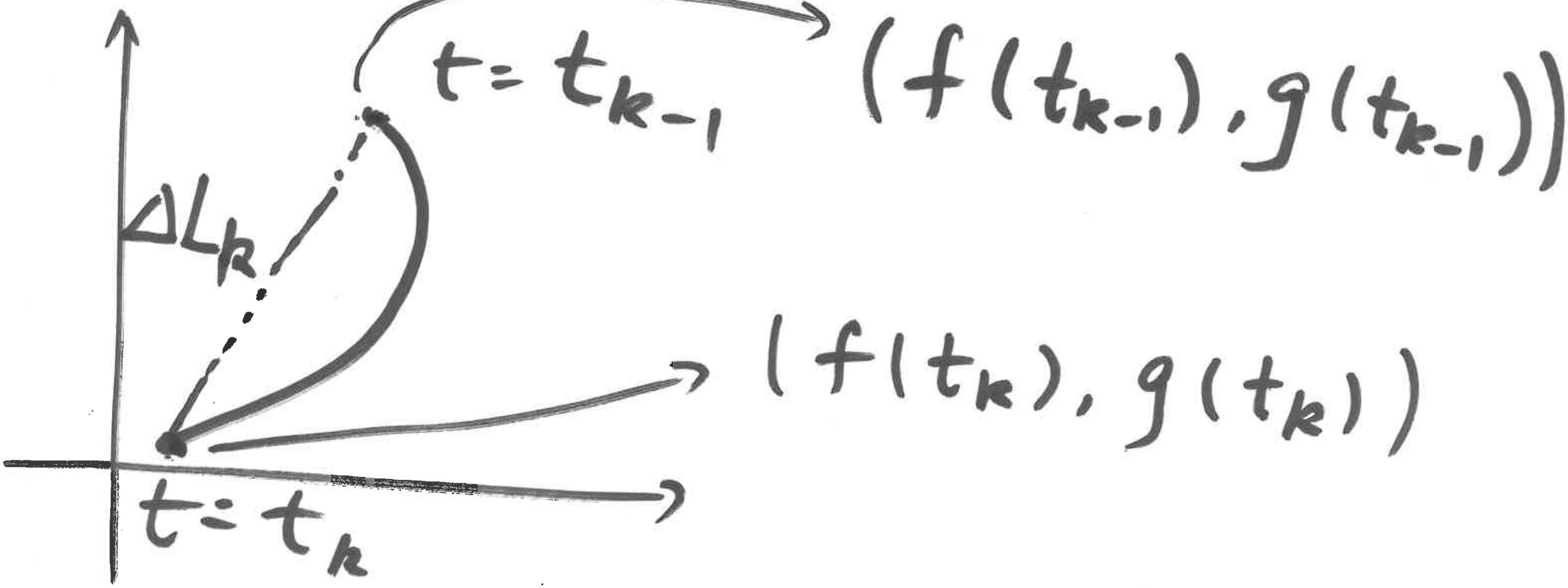
6.8 Length of parametric curve

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

$$a \le t \le b$$

$$t_0 < \cdots < t_{k-1} < t_k < \cdots t_n$$





$$\Delta L_k = \sqrt{(f(t_k) - f(t_{k-1}))^2 + (g(t_k) - g(t_{k-1})}$$

$$f(t_k) - f(t_{k-1})$$

$$= f'(t_k') \cdot \Delta t_k \qquad \Delta t_k = t_k - t_{k-1}$$

$$t_{k-1} \leq t_k' \leq t_k \qquad \underline{\text{Mean value thm}}$$

$$g(t_k) - g(t_{k-1})$$

$$= g'(t_k'') \cdot \Delta t_k \qquad t_{k-1} \leq t_k'' \leq t_k .$$

$$\Delta L_k = \sqrt{(f'(t_k') \cdot \Delta t_k)^2 + (g'(t_k'') \Delta t_k)^2}$$

$$= \Delta t_k \cdot \sqrt{f'(t_k')^2 + g'(t_k'')^2}$$

$$\boxed{L = \int_a^b \underbrace{\sqrt{f'(t)^2 + g'(t)^2}}_{\text{speed}} \, dt}$$

- Special case: graph of $y = g(x)$
  $a \le x \le b$

$$\begin{cases} x = t \\ y = g(t) \end{cases} \quad a \le t \le b$$

$f(t) = t \qquad f'(t) = 1$

$$\boxed{L = \int_a^b \sqrt{1 + g'(t)^2}\, dt}$$

ex 1

$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases} \qquad r > 0, \quad 0 \le t \le 2\pi$$

$f(t) = r\cos t \qquad f'(t) = -r\sin t$

$g(t) = r\sin t \qquad g'(t) = r\cos t$

$$L = \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{r^2}\, dt = \int_0^{2\pi} r\, dt = \boxed{2\pi r}$$

speed $= r$

ex 2

$$\begin{cases} x = t+2 \\ y = 2t+1 \end{cases} \qquad -1 \le t \le 1$$

line segment

$f(t) = t+2 \qquad f'(t) = 1$

$g(t) = 2t+1 \qquad g'(t) = 2$

$$L = \int_{-1}^{1} \sqrt{1^2 + 2^2}\, dt = \boxed{2\sqrt{5}}$$

endpoints: $(1,-1)$, $(3,3)$

$$L = \sqrt{(3-1)^2 + (3-(-1))^2} = \sqrt{20} = 2\sqrt{5}$$

ex 3

$$\begin{cases} x = \dfrac{2t}{1+t^2} \\ y = \dfrac{1-t^2}{1+t^2} \end{cases}$$

$$f(t) = \frac{2t}{1+t^2}$$

$$f'(t) = \frac{2\cdot(1+t^2) - 2t\cdot 2t}{(1+t^2)^2} = \frac{2-2t^2}{(1+t^2)^2}$$

$$g(t) = \frac{1-t^2}{1+t^2}$$

$$g'(t) = \frac{-2t(1+t^2) - (1-t^2)\cdot 2t}{(1+t^2)^2}$$

$$= \frac{-4t}{(1+t^2)^2}$$

$$\sqrt{f'(t)^2 + g'(t)^2} = \sqrt{\left(\frac{2-2t^2}{(1+t^2)^2}\right)^2 + \left(\frac{-4t}{(1+t^2)^2}\right)^2}$$

$$= \sqrt{\frac{4 - 8t^2 + 4t^4 + 16t^2}{(1+t^2)^4}}$$

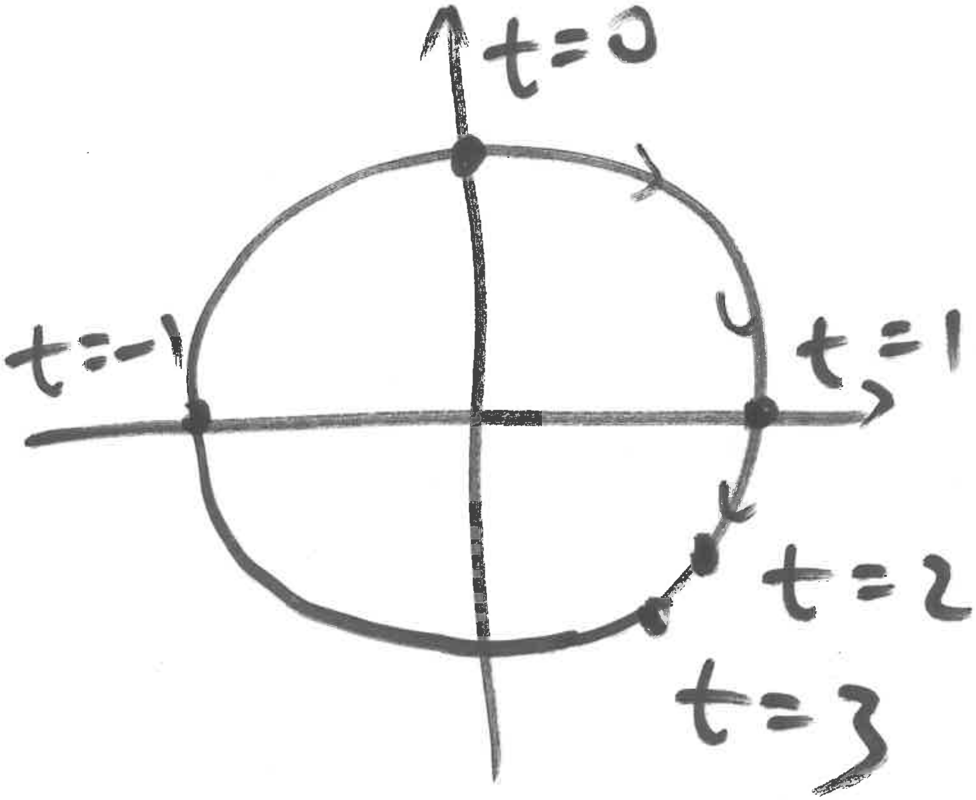
$$4 + 8t^2 + 4t^4$$

$$4(1+2t^2+t^4)$$

$$4(1+t^2)^2$$

$$= \sqrt{\frac{4}{(1+t^2)^2}}$$

$$= \frac{2}{1+t^2} \longleftarrow \text{speed}$$


t=0
t=-1
t=1
t=2
t=3

$$L = \int_{-\infty}^{\infty} \frac{2}{1+t^2}\,dt$$

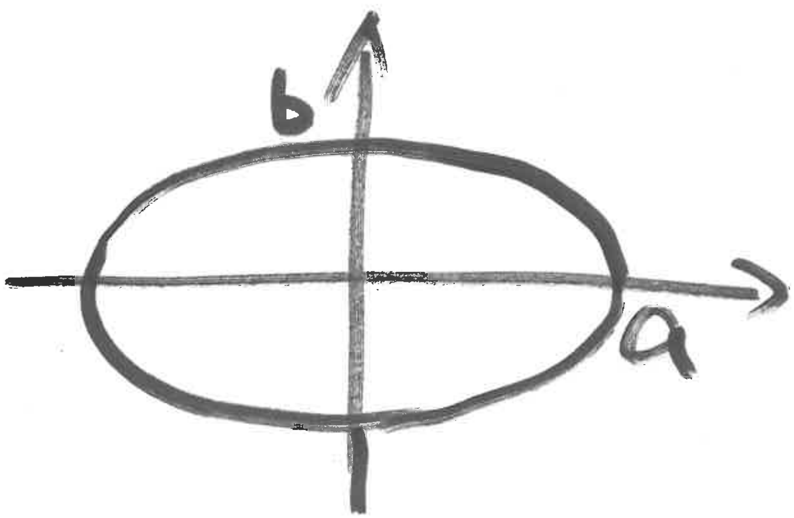
$$= 2 \arctan t \Big|_{-\infty}^{\infty}$$

$$= 2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \boxed{2\pi}$$

ex 4

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \qquad a, b > 0. \qquad 0 \le t \le 2\pi$$

Ellipse



$f(t) = a\cos t$

$f'(t) = -a\sin t$

$g(t) = b\sin t$

$g'(t) = b\cos t$

$$L = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt$$

call it $C_{a,b}$

$a_1 > a_2 \qquad b_1 > b_2$

$C_{a_1,b_1} > C_{a_2,b_2}$

$$C_{a_1,b_1} = \int_0^{2\pi} \sqrt{a_1^2 \sin^2 t + b_1^2 \cos^2 t} \, dt$$

$$C_{a_2,b_2} = \int_0^{2\pi} \sqrt{a_2^2 \sin^2 t + b_2^2 \cos^2 t} \, dt$$

$$a_1 > a_2 > 0 \implies a_1^2 > a_2^2$$

$$b_1^2 > b_2^2$$

$$\sqrt{a_1^2 \sin^2 t + b_1^2 \cos^2 t} > \sqrt{a_2^2 \sin^2 t + b_2^2 \cos^2 t}$$

$$\implies C_{a_1,b_1} > C_{a_2,b_2}.$$